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On the free Carnot $(2, 3, 5, 8)$ group

Abstract. We consider the free nilpotent Lie algebra L with 2 generators, of step 4, and the corresponding connected simply connected Lie group G, with the aim to study the left-invariant sub-Riemannian structure on G defined by the generators of L as an orthonormal frame.

We compute two vector field models of L by polynomial vector fields in $\mathbb{R}^8$, and find an infinitesimal symmetry of the sub-Riemannian structure. Further, we compute explicitly the product rule in G and the right-invariant frame on G.

Key Words and Phrases: Sub-Riemannian geometry, Carnot group.

2010 *Mathematics Subject Classification*: 53C17

Introduction

In this work we start to study a variational problem that can be stated equivalently in the following three ways.

(1) *Geometric statement.* Consider two points $a_0, a_1 \in \mathbb{R}^2$ connected by a smooth curve $\gamma_0 \subset \mathbb{R}^2$. Fix arbitrary data $S \in \mathbb{R}$, $c = (c_x, c_y) \in \mathbb{R}^2$, $M = (M_{xx}, M_{xy}, M_{yy}) \in \mathbb{R}^3$. The problem is to connect the points a_0, a_1 by the shortest smooth curve $\gamma \subset \mathbb{R}^2$ such that the domain $D \subset \mathbb{R}^2$ bounded by $\gamma_0 \cup \gamma$ satisfy the following properties:

(1) $\operatorname{area}(D) = S$,
(2) center of $\operatorname{mass}(D) = c$,
(3) second order $\operatorname{moments}(D) = M$.

(2) *Algebraic statement.* Let L be the free nilpotent Lie algebra with two generators X_1, X_2 of step 4:

(1) $L = \operatorname{span}(X_1, \dots, X_8)$,
(2) $[X_1, X_2] = X_3$,
(3) $[X_1, X_3] = X_4$, $[X_2, X_3] = X_5$,
(4) $[X_1, X_4] = X_6$, $[X_1, X_5] = [X_2, X_4] = X_7$, $[X_2, X_5] = X_8$.

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Let G be the connected simply connected Lie group with the Lie algebra L, we consider $X_1, \dots, X_8$ as a frame of left-invariant vector fields on G. Consider the left-invariant sub-Riemannian structure (G, Δ, g) defined by X_1, X_2 as an orthonormal frame:

$$\Delta_q = \operatorname{span}(X_1(q), X_2(q)), \qquad g(X_i, X_j) = \delta_{ij}.$$

The problem is to find sub-Riemannian length minimizers that connect two given points $q_0, q_1 \in G$:

$$\begin{aligned} &q(t) \in G, \qquad q(0) = q_0, \quad q(t_1) = q_1, \\ &\dot q(t) \in \Delta_{q(t)}, \\ &l = \int_0^{t_1} \sqrt{g(\dot q, \dot q)}\, dt \to \min . \end{aligned}$$

(3) *Optimal control statement.* Consider the following vector fields X_1, X_2 on $\mathbb{R}^8$:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_5} - \frac{x_1 x_2^2}{4}\frac{\partial}{\partial x_7} - \frac{x_2^3}{6}\frac{\partial}{\partial x_8}, \\ X_2 &= \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_4} + \frac{x_1^3}{6}\frac{\partial}{\partial x_6} + \frac{x_1^2 x_2}{4}\frac{\partial}{\partial x_7}. \end{aligned}$$

Given arbitrary points $q_0, q_1 \in \mathbb{R}^8$, it is required to find solutions of the optimal control problem

$$\dot q = u_1 X_1(q) + u_2 X_2(q), \qquad q \in \mathbb{R}^8, \quad (u_1, u_2) \in \mathbb{R}^2, \tag{1}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{2}$$

$$J = \frac{1}{2}\int_0^{t_1} (u_1^2 + u_2^2)\, dt \to \min . \tag{3}$$

The problem stated will be called the nilpotent sub-Riemannian problem with the growth vector $(2, 3, 5, 8)$, or just the $(2, 3, 5, 8)$-problem. There are several important motivations for the study of this problem:

- this problem is a nilpotent approximation of a general sub-Riemannian problem with the growth vector (2,3,5,8) [**1**–**5**];
- this problem is a natural continuation of the important sub-Riemannian (SR) problems: the nilpotent SR problem on the Heisenberg group (aka Dido's problem, growth vector (2,3)) [**6**, **7**], and the nilpotent SR problem on the Cartan group (aka generalized Dido's problem, growth vector (2,3,5)) [**8**–**11**];

- this problem is included into a natural infinite chain of rank 2 SR problems with the free nilpotent Lie algebras of step r, $r \in \mathbb{N}$, and more generally into a natural 2-dimensional lattice of rank d SR problems with the free nilpotent Lie algebras of step r, $(d, r) \in \mathbb{N}^2$;
- this problem is the simplest possible SR problem on a step 4 Carnot group, and it is the first SR problem with growth vector of length 4 that should be studied.

To the best of our knowledge, this is the first study of the (2,3,5,8)-problem (although, it was mentioned in [**12**] as a SR problem with smooth abnormal minimizers).

The structure of this work is as follows.

In Sec. 1 we construct two models ("asymmetric" and "symmetric") of the free nilpotent Lie algebra with 2 generators of step 4 by polynomial vector fields in $\mathbb{R}^8$. For these models, we use respectively an algorithm due to Grayson and Grossman [**13**] and an original approach. In the symmetric model, a one-parameter group of symmetries leaving the initial point fixed is found.

In Sec. 2 we describe explicitly the product rule in the Lie group $G \cong \mathbb{R}^8$, construct a right-invariant frame on G corresponding naturally to the left-invariant frame given by X_1, X_2 and their iterated Lie brackets, compute the corresponding left-invariant and right-invariant Hamiltonians that are linear on fibers of T^*G.

In Conclusion we suggest possible questions for further study.

Results of Sec. 1 of this paper appeared previously in preprint [**14**]. Since both results and techniques of the preprint are necessary for understanding and verification of results of Sec. 2, these materials are published completely in this work.

1. Realisation by polynomial vector fields in $\mathbb{R}^8$

In this section we construct two models of the free nilpotent Lie algebra L(1)–(4) by polynomial vector fields in $\mathbb{R}^8$.

1.1. Free nilpotent Lie algebras

Let $\mathcal{L}_d$ be the real free Lie algebra with d generators [**15**]; $\mathcal{L}_d$ is the Lie algebra of commutators of d variables. We have $\mathcal{L}_d = \oplus_{i=1}^{\infty} \mathcal{L}_d^i$, where $\mathcal{L}_d^i$ is the space of commutator polynomials of degree i. Then $\mathcal{L}_d^{(r)} := \mathcal{L}_d / \oplus_{i=r+1}^{\infty} \mathcal{L}_d^i$ is the free nilpotent Lie algebra with d generators of step r.

Denote $l_d(i) := \dim \mathcal{L}_d^i$, $l_d^{(r)} := \dim \mathcal{L}_d^{(r)} = \sum_{i=1}^r l_d(i)$. The classical expression of $l_d(i)$ is $il_d(i) = d^i - \sum_{j|i,\ 1\leq j<i} jl_d(j)$.

In this work we are interested in free nilpotent Lie algebras with 2 generators. Dimensions of such Lie algebras for small step are given in Table 1.

TABLE 1. Dimensions of free nilpotent Lie algebras $\mathcal{L}_2^{(i)}$

i	1	2	3	4	5	6	7	8	9	10
$l_2(i)$	2	1	2	3	6	9	18	30	56	99
$l_2^{(i)}$	2	3	5	8	14	23	41	71	127	226

1.2. Carnot algebras and groups

A Lie algebra L is called a Carnot algebra if it admits a decomposition $L = \oplus_{i=1}^r L_i$ as a vector space, such that $[L_i, L_j] \subset L_{i+j}$, $L_s = 0$ for $s > r$, $L_{i+1} = [L_1, L_i]$.

A free nilpotent Lie algebra $\mathcal{L}_d^{(r)}$ is a Carnot algebra with the homogeneous components $L_i = \mathcal{L}_d^i$.

A Carnot group G is a connected, simply connected Lie group whose Lie algebra L is a Carnot algebra. If L is realized as the Lie algebra of left-invariant vector fields on G, then the degree 1 component L_1 can be thought of as a completely nonholonomic (bracket-generating) distribution on G. If moreover L_1 is endowed with a left-invariant inner product g, then (G, L_1, g) becomes a nilpotent left-invariant sub-Riemannian manifold [**5**]. Such sub-Riemannian structures are nilpotent approximations of generic sub-Riemannian structures [**1**–**4**].

The sequence of numbers

$$(\dim L_1, \dim L_1 + \dim L_2, \dots, \dim L_1 + \cdots + \dim L_r = \dim L)$$

is called the growth vector of the distribution L_1 [**7**].

For free nilpotent Lie algebras, the growth vector is maximal compared with all Carnot algebras with the bidimension $(\dim L_1, \dim L)$.

1.3. Lie algebra with the growth vector $(2, 3, 5, 8)$

The Carnot algebra with the growth vector (2, 3, 5, 8)

$$\mathcal{L}_2^{(4)} = \operatorname{span}(X_1, \dots, X_8)$$

is determined by the following multiplication table:

$$[X_1, X_2] = X_3, \tag{4}$$

$$[X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \tag{5}$$

$$[X_1, X_4] = X_6, \quad [X_1, X_5] = [X_2, X_4] = X_7, \quad [X_2, X_5] = X_8, \tag{6}$$

with all the rest brackets equal to zero. This multiplication table is depicted at Fig. 1.

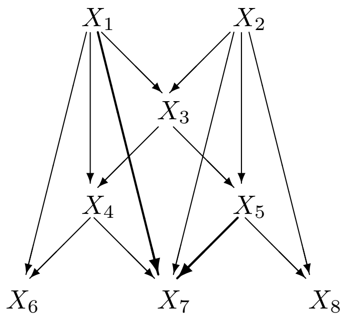

FIGURE 1. Lie algebra with the growth vector $(2,3,5,8)$

1.4. Hall basis

Free nilpotent Lie algebras have a convenient basis introduced by M. Hall [**16**]. We describe it using the exposition of [**13**].

The Hall basis of the free Lie algebra $\mathcal{L}_d$ with d generators $X_1, \dots, X_d$ is the subset Hall $\subset \mathcal{L}_d$ that has a decomposition into homogeneous components $\mathrm{Hall} = \cup_{i=1}^{\infty} \mathrm{Hall}_i$ defined as follows.

Each element H_j, $j = 1, 2, \dots,$ of the Hall basis is a monomial in the generators X_i and is defined recursively as follows. The generators satisfy the inclusion $X_i \in \mathrm{Hall}_1$, $i = 1, \dots, d$, and we denote $H_i = X_i$, $i = 1, \dots, d$. If we have defined basis elements $H_1, \dots, H_{N_{p-1}} \in \oplus_{j=1}^{p-1} \mathrm{Hall}_j$, they are simply ordered so that $E < F$ if $E \in \mathrm{Hall}_k$, $F \in \mathrm{Hall}_l$, $k < l$: $H_1 < H_2 < \cdots < H_{N_{p-1}}$. Also if $E \in \mathrm{Hall}_s$, $F \in \mathrm{Hall}_t$ and $p = s + t$, then $[E, F] \in \mathrm{Hall}_p$ if:

(1) $E > F$, and
(2) if $E = [G, K]$, then $K \in \mathrm{Hall}_q$ and $t \geq q$.

By this definition, one easily computes recursively the first components Hall_i of the Hall basis for $d = 2$:

$\mathrm{Hall}_1 = \{H_1, H_2\}, \quad H_1 = X_1, \quad H_2 = X_2,$
$\mathrm{Hall}_2 = \{H_3\}, \quad H_3 = [X_2, X_1],$
$\mathrm{Hall}_3 = \{H_4, H_5\}, \quad H_4 = [[X_2, X_1], X_1], \quad H_5 = [[X_2, X_1], X_2],$
$\mathrm{Hall}_4 = \{H_6, H_7, H_8\},$
$H_6 = [[[X_2, X_1], X_1], X_1], \quad H_7 = [[[X_2, X_1], X_1], X_2],$
$H_8 = [[[X_2, X_1], X_2], X_2].$

Consequently, $\mathcal{L}_2^{(4)} = \mathrm{span}\{H_1, \dots, H_8\}$. In the sequel we use a more convenient basis of $\mathcal{L}_2^{(4)} = \mathrm{span}\{X_1, \dots, X_8\}$ with the multiplication table (4)–(6).

1.5. Asymmetric vector field model for $\mathcal{L}_2^{(4)}$

Here we recall an algorithm for construction of a vector field model for the Lie algebra $\mathcal{L}_2^{(r)}$ due to Grayson and Grossman [13]. For a given $r \geq 1$, the algorithm evaluates two polynomial vector fields $H_1, H_2 \in \mathrm{Vec}(\mathbb{R}^N)$, $N = \dim \mathcal{L}_2^{(r)}$, which generate the Lie algebra $\mathcal{L}_2^{(r)}$.

Consider the Hall basis elements $\mathrm{span}\{H_1, \dots, H_N\} = \mathcal{L}_2^{(r)}$. Each element $H_i \in \mathrm{Hall}_j$ is a Lie bracket of length j:

$H_i = [\dots[[H_2, H_{k_j}], H_{k_{j-1}}], \dots, H_{k_1}]$, $k_j = 1$, $k_{n+1} \leq k_n$ for $1 \leq n \leq j-1$.

This defines a partial ordering of the basis elements. We say that H_i is a direct descendant of H_2 and of each H_{k_l} and write $i \succ 2$, $i \succ k_l$, $l = 1, \dots, j$.

Define monomials $P_{2,k}$ in $x_1, \dots, x_N$ inductively by

$$P_{2,k} = -x_j \; P_{2,i}/(\deg_j P_{2,i} + 1),$$

whenever $H_k = [H_i, H_j]$ is a basis Hall element, and where $\deg_j P$ is the highest power of x_j which divides P.

The following theorem gives the properties of the generators.

Theorem 1 (Th. 3.1 [13]). *Let $r \geq 1$ and let $N = \dim \mathcal{L}_2^{(r)}$. Then the vector fields $H_1 = \dfrac{\partial}{\partial x_1}$, $H_2 = \dfrac{\partial}{\partial x_2} + \sum_{i \succ 2} P_{2,i} \dfrac{\partial}{\partial x_i}$ have the following properties:*

(1) *they are homogeneous of weight one with respect to the grading*

$$\mathbb{R}^N = \mathrm{Hall}_1 \oplus \dots \oplus \mathrm{Hall}_r;$$

(2) $\mathrm{Lie}(H_1, H_2) = \mathcal{L}_2^{(r)}$.

The algorithm described before Theorem 1 produces the following vector field basis of $\mathcal{L}_2^{(4)}$:

$$H_1 = \frac{\partial}{\partial x_1},$$
$$H_2 = \frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3} - \frac{x_1^2}{2}\frac{\partial}{\partial x_4} - x_1x_2\frac{\partial}{\partial x_5} + \frac{x_1^3}{6}\frac{\partial}{\partial x_6} + \frac{x_1^2x_2}{2}\frac{\partial}{\partial x_7} + \frac{x_1x_2^2}{2}\frac{\partial}{\partial x_8},$$
$$H_3 = \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_5} - \frac{x_1^2}{2}\frac{\partial}{\partial x_6} - x_1x_2\frac{\partial}{\partial x_7} - \frac{x_2^2}{2}\frac{\partial}{\partial x_8},$$
$$H_4 = -\frac{\partial}{\partial x_4} + x_1\frac{\partial}{\partial x_6} + x_2\frac{\partial}{\partial x_7},$$
$$H_5 = -\frac{\partial}{\partial x_5} + x_1\frac{\partial}{\partial x_7} + x_2\frac{\partial}{\partial x_8},$$
$$H_6 = -\frac{\partial}{\partial x_6}, \quad H_7 = -\frac{\partial}{\partial x_7}, \quad H_8 = -\frac{\partial}{\partial x_8},$$

with the multiplication table

$$[H_2, H_1] = H_3, \tag{7}$$
$$[H_3, H_1] = H_4,\ [H_3, H_2] = H_5, \tag{8}$$
$$[H_4, H_1] = H_6,\ [H_4, H_2] = H_7,\ [H_5, H_2] = H_8. \tag{9}$$

1.6. Symmetric vector field model of $\mathcal{L}_2^{(4)}$

The vector field model of the Lie algebra $\mathcal{L}_2^{(4)}$ via the fields $H_1, \ldots, H_8$ obtained in the previous subsection is asymmetric in the sense that there is no visible symmetry between the vector fields H_1 and H_2. Moreover, no continuous symmetries of the sub-Riemannian structure generated by the orthonormal frame $\{H_1, H_2\}$ are visible, although the Lie brackets (7)–(9) suggest that this sub-Riemannian structure should be preserved by a one-parameter group of rotations in the plane $\operatorname{span}\{H_1, H_2\}$.

One can find a symmetric vector field model of $\mathcal{L}_2^{(4)}$ free of such shortages as in the following statement.

Theorem 2. (1) *The vector fields*

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_5} - \frac{x_1x_2^2}{4}\frac{\partial}{\partial x_7} - \frac{x_2^3}{6}\frac{\partial}{\partial x_8}, \tag{10}$$
$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_4} + \frac{x_1^3}{6}\frac{\partial}{\partial x_6} + \frac{x_1^2x_2}{4}\frac{\partial}{\partial x_7}, \tag{11}$$

(12) $$X_3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + \frac{x_1^2}{2} \frac{\partial}{\partial x_6} + x_1 x_2 \frac{\partial}{\partial x_7} + \frac{x_2^2}{2} \frac{\partial}{\partial x_8},$$

(13) $$X_4 = \frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_6} + x_2 \frac{\partial}{\partial x_7},$$

(14) $$X_5 = \frac{\partial}{\partial x_5} + x_1 \frac{\partial}{\partial x_7} + x_2 \frac{\partial}{\partial x_8},$$

(15) $$X_6 = \frac{\partial}{\partial x_6},$$

(16) $$X_7 = \frac{\partial}{\partial x_7},$$

(17) $$X_8 = \frac{\partial}{\partial x_8}$$

satisfy the multiplication table (4)–(6). *Thus the fields* $X_1, \dots, X_8 \in \mathrm{Vec}(\mathbb{R}^8)$ *model the Lie algebra* $\mathcal{L}_2^{(4)}$.

(2) *The vector field*

(18)
$$X_0 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_5} + P \frac{\partial}{\partial x_6} + Q \frac{\partial}{\partial x_7} + R \frac{\partial}{\partial x_8},$$

(19)
$$P = -\frac{x_1^4}{24} + \frac{x_1^2 x_2^2}{8} + x_7,$$

(20)
$$Q = \frac{x_1 x_2^3}{12} + \frac{x_1^3 x_2}{12} - 2x_6 + 2x_8,$$

(21)
$$R = \frac{x_1^2 x_2^2}{8} - \frac{x_2^4}{24} - x_7$$

satisfies the following relations:

(22) $$[X_0, X_1] = X_2, \qquad [X_0, X_2] = -X_1, \qquad [X_0, X_3] = 0,$$

(23) $$[X_0, X_4] = X_5, \qquad [X_0, X_5] = -X_4,$$

(24) $$[X_0, X_6] = 2X_7, \qquad [X_0, X_7] = X_8 - X_6, \qquad [X_0, X_8] = -2X_7.$$

Thus the field X_0 *is an infinitesimal symmetry of the sub-Riemannian structure generated by the orthonormal frame* $\{X_1, X_2\}$.

Proof. In fact, both statements of the proposition are verified by direct computation, but we prefer to describe a method of construction of the vector fields $X_1, \dots, X_8$, and X_0.

(1) In the previous work [8] we constructed a similar symmetric vector field model for the Lie algebra $\mathcal{L}_2^{(3)}$, which has growth vector (2, 3, 5):

$$\mathcal{L}_2^{(3)} = \operatorname{span}\{X_1, \dots, X_5\} \subset \operatorname{Vec}(\mathbb{R}^5), \tag{25}$$

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_5}, \tag{26}$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_4}, \tag{27}$$

$$X_3 = \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_5}, \tag{28}$$

$$X_4 = \frac{\partial}{\partial x_4}, \tag{29}$$

$$X_5 = \frac{\partial}{\partial x_5}, \tag{30}$$

with the Lie brackets (4), (5). Now we aim to "continue" these relationships to vector fields $X_1, \dots, X_8 \in \operatorname{Vec}(\mathbb{R}^8)$ that span the Lie algebra $\mathcal{L}_2^{(4)}$. So we seek for vector fields of the form

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_5} + \sum_{i=6}^{8} a_1^i \frac{\partial}{\partial x_i}, \tag{31}$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_4} + \sum_{i=6}^{8} a_2^i \frac{\partial}{\partial x_i}, \tag{32}$$

$$X_3 = \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_5} + \sum_{i=6}^{8} a_3^i \frac{\partial}{\partial x_i}, \tag{33}$$

$$X_4 = \frac{\partial}{\partial x_4} + \sum_{i=6}^{8} a_4^i \frac{\partial}{\partial x_i}, \tag{34}$$

$$X_5 = \frac{\partial}{\partial x_5} + \sum_{i=6}^{8} a_5^i \frac{\partial}{\partial x_i}, \tag{35}$$

$$X_j = \sum_{i=6}^{8} a_i^j \frac{\partial}{\partial x_j}, \qquad j = 6, 7, 8, \tag{36}$$

such that $\operatorname{span}\{X_1,\ldots,X_8\}=\mathcal{L}_2^{(4)}$.

Compute the required Lie brackets:

$$\begin{aligned}
[X_1,X_2] &= \frac{\partial}{\partial x_3}+x_1\frac{\partial}{\partial x_4}+x_2\frac{\partial}{\partial x_5}+\left(\frac{\partial a_2^6}{\partial x_1}-\frac{\partial a_1^6}{\partial x_2}\right)\frac{\partial}{\partial x_6}\\
&\quad+\left(\frac{\partial a_2^7}{\partial x_1}-\frac{\partial a_1^7}{\partial x_2}\right)\frac{\partial}{\partial x_7}+\left(\frac{\partial a_2^8}{\partial x_1}-\frac{\partial a_1^8}{\partial x_2}\right)\frac{\partial}{\partial x_8},\\
[X_1,X_3] &= \frac{\partial}{\partial x_4}+\frac{\partial a_3^6}{\partial x_1}\frac{\partial}{\partial x_6}+\frac{\partial a_3^7}{\partial x_1}\frac{\partial}{\partial x_7}+\frac{\partial a_3^8}{\partial x_1}\frac{\partial}{\partial x_8},\\
[X_2,X_3] &= \frac{\partial}{\partial x_5}+\frac{\partial a_3^6}{\partial x_2}\frac{\partial}{\partial x_6}+\frac{\partial a_3^7}{\partial x_2}\frac{\partial}{\partial x_7}+\frac{\partial a_3^8}{\partial x_2}\frac{\partial}{\partial x_8},\\
[X_1,X_4] &= \frac{\partial a_4^6}{\partial x_1}\frac{\partial}{\partial x_6}+\frac{\partial a_4^7}{\partial x_1}\frac{\partial}{\partial x_7}+\frac{\partial a_4^8}{\partial x_1}\frac{\partial}{\partial x_8},\\
[X_1,X_5] &= \frac{\partial a_5^6}{\partial x_1}\frac{\partial}{\partial x_6}+\frac{\partial a_5^7}{\partial x_1}\frac{\partial}{\partial x_7}+\frac{\partial a_5^8}{\partial x_1}\frac{\partial}{\partial x_8},\\
[X_2,X_4] &= \frac{\partial a_4^6}{\partial x_2}\frac{\partial}{\partial x_6}+\frac{\partial a_4^7}{\partial x_2}\frac{\partial}{\partial x_7}+\frac{\partial a_4^8}{\partial x_2}\frac{\partial}{\partial x_8},\\
[X_2,X_5] &= \frac{\partial a_5^6}{\partial x_2}\frac{\partial}{\partial x_6}+\frac{\partial a_5^7}{\partial x_2}\frac{\partial}{\partial x_7}+\frac{\partial a_5^8}{\partial x_2}\frac{\partial}{\partial x_8}.
\end{aligned}$$

The vector fields $X_1,\ldots,X_8$ should be independent, thus the determinant constructed of these vectors as columns should satisfy the inequality

$$D=\det(X_1,\ldots,X_8)=\begin{vmatrix} a_6^6 & a_7^6 & a_8^6\\ a_6^7 & a_7^7 & a_8^7\\ a_6^8 & a_7^8 & a_8^8\end{vmatrix}\neq 0.$$

We will choose a_i^j such that $D=1$. It follows from the multiplication table for $X_1,\ldots,X_8$ that

$$D=\begin{vmatrix} \dfrac{d^2a_3^6}{dx_1^2} & \dfrac{d^2a_3^6}{dx_1dx_2} & \dfrac{d^2a_3^6}{dx_2^2}\\ \dfrac{d^2a_3^7}{dx_1^2} & \dfrac{d^2a_3^7}{dx_1dx_2} & \dfrac{d^2a_3^7}{dx_2^2}\\ \dfrac{d^2a_3^8}{dx_1^2} & \dfrac{d^2a_3^8}{dx_1dx_2} & \dfrac{d^2a_3^8}{dx_2^2}\end{vmatrix}.$$

In order to get $D = 1$, define the entries of this matrix in the following symmetric way: $a_3^6 = \frac{x_1^2}{2}$, $a_3^7 = x_1 x_2$, $a_3^8 = \frac{x_2^2}{2}$. Then we obtain from the multiplication table for $X_1, \dots, X_8$ that $\frac{\partial a_2^6}{\partial x_1} - \frac{\partial a_1^6}{\partial x_2} = a_3^6 = \frac{x_1^2}{2}$, $\frac{\partial a_2^7}{\partial x_1} - \frac{\partial a_1^7}{\partial x_2} = a_3^7 = x_1 x_2$, $\frac{\partial a_2^8}{\partial x_1} - \frac{\partial a_1^8}{\partial x_2} = a_3^8 = \frac{x_2^2}{2}$. We solve these equations in the following symmetric way: $a_1^6 = 0$, $a_2^6 = \frac{x_1^3}{6}$, $a_1^7 = -\frac{x_1 x_2^2}{4}$, $a_2^7 = \frac{x_1^2 x_2}{4}$, $a_1^8 = -\frac{x_2^3}{6}$, $a_2^8 = 0$. Then we substitute these coefficients to (31), (32) and check item (1) of this theorem by direct computation.

Now we prove item (2). We proceed exactly as for item (1): we start from an infinitesimal symmetry [8]

$$X_0 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_5} \in \mathrm{Vec}(\mathbb{R}^5) \tag{37}$$

of the sub-Riemannian structure on $\mathbb{R}^5$ determined by the orthonormal frame (26), (27) and "continue" symmetry (37) to the sub-Riemannian structure on $\mathbb{R}^8$ determined by the orthonormal frame (10), (11).

So we seek for a vector field $X_0 \in \mathrm{Vec}(\mathbb{R}^8)$ of the form (18) for the functions $P, Q, R \in C^\infty(\mathbb{R}^8)$ to be determined so that the multiplication table (22)–(24) hold.

The first two equalities in (22) yield $X_1 P = -\frac{x_1^3}{6}$, $X_2 P = \frac{x_1^2 x_2}{2}$. Further, $X_3 P = [X_1, X_2] P = X_1 X_2 P - X_2 X_1 P = X_1 \frac{x_1^2 x_2}{2} + X_2 \frac{x_1^3}{6} = x_1 x_2$. Similarly it follows that $X_4 P = x_2$, $X_5 P = x_1$, $X_6 P = 0$, $X_7 P = 1$, $X_8 P = 0$. Since $X_6 P = X_8 P = 0$, then $P = P(x_1, x_2, x_3, x_4, x_5, x_7)$. Moreover, since $X_7 P = 1$, then $P = x_7 + a(x_1, x_2, x_3, x_4, x_5)$. The equality $X_5 P = x_1$ implies that $\frac{\partial a}{\partial x_5} = 0$, i.e., $a = a(x_1, x_2, x_3, x_4)$. Similarly, since $X_4 P = x_2$, then $a = a(x_1, x_2, x_3)$. It follows from the equality $X_3 P = x_1 x_2$ that $\frac{\partial a}{\partial x_3} = x_1 x_2$, i.e., $a = x_1 x_2 x_3 + b(x_1, x_2)$. Moreover, the equality $X_2 P = \frac{x_1^2 x_2}{2}$ implies that $\frac{\partial b}{\partial x_2} = -x_1 x_3 - \frac{x_1^2 x_2}{4}$, i.e., $b = -x_1 x_2 x_3 - \frac{x_1^2 x_2^2}{8} + c(x_1)$. Finally, the equality $X_1 P = -\frac{x_1^3}{2}$ implies that $\frac{d\,c}{d\,x_1} = -\frac{x_1^3}{6} + \frac{x_1 x_2^2}{2}$ i.e., $c = -\frac{x_1^4}{24} + \frac{x_1^2 x_2^2}{4}$. Thus equality (19) follows.

Similarly we get equalities (20), (21).

Then multiplication table (22)–(24) for the vector field (18)–(21) is verified by a direct computation. □

2. Carnot group

In this section we study the Carnot group G with the Lie algebra $L = \mathcal{L}_2^{(4)}$.

2.1. Product rule in G

In this subsection we compute the product rule in the connected simply connected Lie group G with the Lie algebra $L = \mathcal{L}_2^{(4)}$ on which the vector fields $X_1, \dots, X_8$ given by (10)–(17) are left-invariant.

Our algorithm for computation of the product rule in a Lie group G with a known left-invariant frame $X_1, \dots, X_n \in \mathrm{Vec}(G)$ follows from the next argument. Let $g_1, g_2 \in G$, and let $g_2 = e^{t_n X_n} \circ \dots \circ e^{t_1 X_1}(\mathrm{Id})$, $t_1, \dots, t_n \in \mathbb{R}$, where we denote by $e^{tX} : G \to G$ the flow of the vector field X. Then $g_1 \cdot g_2 = g_1 \cdot e^{t_n X_n} \circ \dots \circ e^{t_1 X_1}(\mathrm{Id}) = e^{t_n X_n} \circ \dots \circ e^{t_1 X_1}(g_1)$ by left-invariance of X_i. So an algorithm for computation of $g_1 \cdot g_2$ is the following:

(1) Compute $e^{t_i X_i}(g), \quad t_i \in \mathbb{R},\ g \in G$.
(2) Compute $e^{t_n X_n} \circ \dots \circ e^{t_1 X_1}(g), \quad t_i \in \mathbb{R},\ g \in G$.
(3) Solve the equation $e^{t_n X_n} \circ \dots \circ e^{t_1 X_1}(\mathrm{Id}) = g_2$ for $t_1, \dots, t_n \in \mathbb{R}$ (we assume that this is possible in a unique way).
(4) Compute $g_1 \cdot g_2 = e^{t_n X_n} \circ \dots \circ e^{t_1 X_1}(g_2)$.

By this algorithm, we compute the product $z = x \cdot y$ in the coordinates on G (notice that as a manifold $G = \mathbb{R}^8$), as follows:

$$x = (x_1, \dots, x_8),\ y = (y_1, \dots, y_8),\ z = (z_1, \dots, z_8) \in G = \mathbb{R}^8,$$

$$z_1 = x_1 + y_1, \quad z_2 = x_2 + y_2,$$

$$z_3 = x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1),$$

$$z_4 = x_4 + y_4 + \frac{1}{2}(x_1(x_1 + y_1) + x_2(x_2 + y_2) + x_1 y_3),$$

$$z_5 = x_5 + y_5 - \frac{1}{2} y_1 (x_1(x_1 + y_1) + x_2(x_2 + y_2)) + x_2 y_3,$$

$$z_6 = x_6 + y_6 + \frac{x_1}{12}(2x_1^2 y_2 + 3x_1 y_1 y_2 - 2y_2^3 + 6x_1 y_3 + 12 y_4),$$

$$z_7 = x_7 + y_7 + \frac{1}{24}(3x_1^2 y_2(2x_2 + y_2) - x_2(3x_2 y_1^2 + 6y_1^2 y_2 + 4(y_2^3 - 6y^4)) + x_1(-6x_2^2 y_1 + 4y_1^3 + 6y_1 y_2^2 + 24x_2 y_3 + 24y_5)),$$

$$z_8 = x_8 + y_8 + \frac{x_2}{2}(-2x_2^2 y_1 + 2y_1^3 - 3x_2 y_1 y_2 + 6x_2 y_3 + 12y_5).$$

2.2. Right-invariant frame on G

Computation of the right-invariant frame on G corresponding to a left-invariant frame can be done via the following simple lemma. Denote the inversion on a Lie group G as $i : G \to G$, $i(g) = g^{-1}$.

LEMMA 1. *Let $X_1, X_2, X_3 \in \operatorname{Vec}(G)$ and $Y_1, Y_2, Y_3 \in \operatorname{Vec}(G)$ be respectively left-invariant and right-invariant vector fields on a Lie group G such that $Y_j(\mathrm{Id}) = -X_j(\mathrm{Id})$, $j = 1, 2, 3$. Then*

$$i_* X_j = Y_j, \quad i = 1, 2, 3, \tag{38}$$

$$[X_1, X_2] = X_3 \quad \Leftrightarrow \quad [Y_1, Y_2] = Y_3. \tag{39}$$

PROOF. Equality (38) follows by the left-invariance and right-invariance of the fields X_i and Y_i respectively. Equality (39) follows since the diffeomorphism $i : G \to G$ preserves Lie bracket of vector fields (see e.g. [**17**]). □

Thus if $X_1, \dots, X_n \in \operatorname{Vec} G$ is a left-invariant frame on a Lie group G, then $Y_1, \dots, Y_n \in \operatorname{Vec} G$, $Y_j = i_* X_j$, is the right-invariant frame such that $Y_j(\mathrm{Id}) = -X_j(\mathrm{Id})$, $j = 1, \dots, n$, and the same product rules as for $X_1, \dots, X_n$.

Immediate computation using the product rule in G given in Subsec. 2.1 gives the following right-invariant frame on the Lie group $G = \mathbb{R}^8$:

$$Y_1 = -\frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1 x_2 + 2x_3}{2}\frac{\partial}{\partial x_4} + \frac{x_1^2}{2}\frac{\partial}{\partial x_5} + \frac{x_2^3 - 6x_4}{6}\frac{\partial}{\partial x_6} - \frac{2x_1^3 + 3x_1 x_2^2 + 12x_5}{12}\frac{\partial}{\partial x_7},$$

$$Y_2 = -\frac{\partial}{\partial x_2} - \frac{x_1}{2}\frac{\partial}{\partial x_3} - \frac{x_2^2}{2}\frac{\partial}{\partial x_4} + \frac{x_1 x_2 - 2x_3}{2}\frac{\partial}{\partial x_5} + \frac{3x_1^2 x_2 + 2x_2^3 - 12x_4}{12}\frac{\partial}{\partial x_6} - \frac{x_1^3 + 6x_5}{6}\frac{\partial}{\partial x_8},$$

$$Y_i = -\frac{\partial}{\partial x_i}, \qquad i = 3, \dots, 8.$$

2.3. Left-invariant and right-invariant Hamiltonians on T^*G

Using the expressions for the left-invariant and right-invariant frames given in Subsec. 1.6 and Subsec. 2.2, we define the corresponding left-invariant and right-invariant Hamiltonians, linear on fibers in T^*G:

$$h_i(\lambda) = \langle \lambda, X_i \rangle, \qquad g_i(\lambda) = \langle \lambda, Y_i \rangle, \qquad \lambda \in T^*G, \quad i = 1, \dots, 8.$$

In the canonical coordinates $(x_1, \dots, x_8, \psi_1, \dots, \psi_8)$ on T^*G [17] we have the following:

$$\begin{aligned}
h_1 &= \psi_1 - \frac{x_2}{2}\psi_3 - \frac{x_1^2 + x_2^2}{2}\psi_5 - \frac{x_1 x_2^2}{4}\psi_7 - \frac{x_2^3}{6}\psi_8, \\
h_2 &= \psi_2 + \frac{x_1}{2}\psi_3 + \frac{x_1^2 + x_2^2}{2}\psi_4 + \frac{x_1^3}{6}\psi_6 + \frac{x_1^2 x_2}{4}\psi_7, \\
h_3 &= \psi_3 + x_1\psi_4 + x_2\psi_5 + \frac{x_1^2}{2}\psi_6 + x_1 x_2 \psi_7 + \frac{x_2^2}{2}\psi_8, \\
h_4 &= \psi_4 + x_1\psi_6 + x_2\psi_7, \\
h_5 &= \psi_5 + x_1\psi_7 + x_2\psi_8, \\
h_i &= \psi_i, \qquad i = 6, 7, 8,
\end{aligned}$$

and

$$\begin{aligned}
g_1 = -\psi_1 - \frac{x_2}{2}\psi_3 - \frac{x_1 x_2 + 2x_3}{2}\psi_4 + \frac{x_1^2}{2}\psi_5 \\
+ \frac{x_2^3 - 6x_4}{6}\psi_6 - \frac{2x_1^3 + 3x_1 x^2 + 12x_5}{12}\psi_7, \qquad (40)
\end{aligned}$$

$$\begin{aligned}
g_2 = -\psi_2 - \frac{x_1}{2}\psi_3 - \frac{x_2^2}{2}\psi_4 + \frac{x_1 x_2 - 2x_3}{2}\psi_5 \\
+ \frac{3x_1^2 x_2 + 2x_2^3 - 12x_4}{12}\psi_6 - \frac{x_1^3 + 6x_5}{6}\psi_8, \qquad (41)
\end{aligned}$$

$$g_i = -\psi_i, \qquad i = 3, \dots, 8. \qquad (42)$$

Conclusion

We see the following interesting questions for the (2,3,5,8)-problem:

(1) study optimality of abnormal geodesics;
(2) describe all cases where the normal Hamiltonian vector field $\vec{H}$ is Liouville intergable, integrate and study the corresponding normal geodesics;
(3) describe precisely the chaotic dynamics of the normal Hamiltonian vector field $\vec{H}$ suggested by numerical simulations.

We plan to address these questions in forthcoming works.

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Ж.-П. Готье, Ю. Л. Сачков. *О свободной группе Карно с вектором роста* $(2,3,5,8)$.

Аннотация. Рассматривается свободная нильпотентная алгебра Ли L с двумя генераторами, ступени 4, и соответствующая связная односвязная группа Ли G, с целью исследования левоинвариантной субримановой структуры на G, заданной генераторами алгебры Ли.

Вычислены две модели алгебры Ли L с помощью векторных полей в $\mathbb{R}^8$, и найдены инфинитезимальные симметрии субримановой структуры. Явно вычислены закон умножения в группе Ли G и правоинвариантный репер на G. (*Англ.*)

Ключевые слова и фразы: Субриманова геометрия, группы Карно.

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