ISSN 2079-3316 PROGRAM SYSTEMS: THEORY AND APPLICATIONS no.4(31), 2016, pp. 231–247

A. Mashtakov, R. Duits

A Cortical Based Model for Contour Completion on the Retinal Sphere

Abstract. We introduce a natural spherical extension of a well-known contour perception model due to J. Petitot, G. Citti and A. Sarti, which additionally takes into account the spherical nature of the retina. Such a spherical extension was initially proposed by U. Boscain and F. Rossi. We extend their model by taking into account a relevant anisotropy parameter controlling the stiffness of optimal spherical contours, and an external cost modeling the non-uniform distribution of photoreceptors on the retina.

Key words and phrases: Association Field, Visual Cortex, Cortical Magnification, Sub-Riemannian Geodesics, Lie Group SO(3).

Introduction

Modeling of a visual system of mammals has been attracting a great interest of many researchers in recent years. The investigation of Hubel and Wiesel [1] (Nobel Prize in Physiology or Medicine, 1981) has produced a strong progress in understanding of the functional architecture of the primary visual cortex V1. Hubel and Wiesel have realized that specific neurons in the visual areas of the cerebral cortex are connected to certain areas of the visual field of the retina. They performed an experiment showing that the neurons in different areas of the visual cortex react to various directions at the same locations in the visual field. It was understood that, for efficient image processing, the brain stores the image not as a sequence of points, but as a sequence of strokes (points and directions tangential to the contour). Thus, a contact structure in the extended space of locations and directions over the retina naturally appears in modelling of V1.

The study was funded by RFBR, research project No. 16-31-60083_mol_a_dk(1).

The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013)/ERC grant Lie Analysis, a.n. 335555(2).

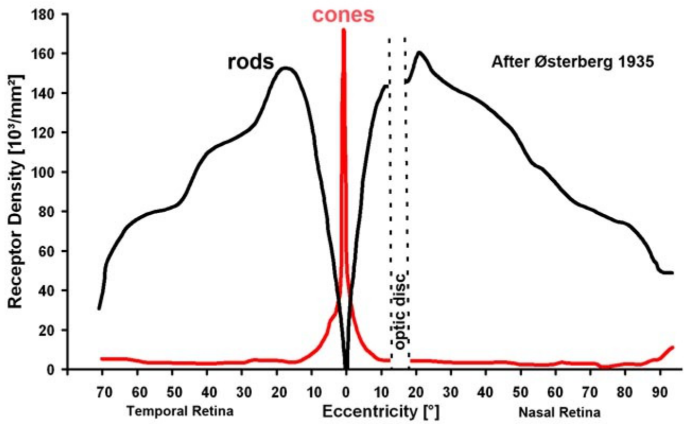


Figure 1: Densities of rods and cones in Human retina

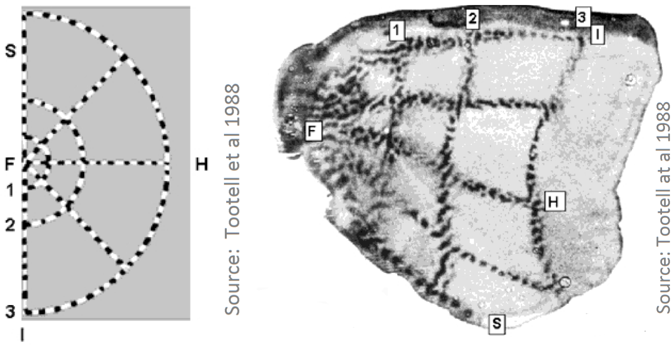
Therefore, Petitot [2], Citti and Sarti [3] proposed to model V1 by a sub-Riemannian contact structure on Lie groups H(3) and SE(2). In their model the retina is represented by a real plane, and sub-Riemannian geodesics arise naturally as the curves that minimize the energy expended to create a connection between the excited neurons.

In this work, inspired by [4], we refine the Petitot-Citti-Sarti model 1) by including the curvature of the retina, inducing an SO(3) Lie group structure (instead of an SE(2) or H(3) group structure);
2) by including an external cost, that accounts for a non-uniform distribution of photoreceptors on the retina [5, 6], see Fig. 1.

In our model, we approximate the retina by a hemisphere and propose a realistic external cost, that performs adaptation to the nonuniform distribution of photoreceptors.

It is known [7, 8], that a small area of the retina responsible for central vision (macula) is processed in a large area of the striate cortex (V1), see Fig. 2. Empirically, this retina-cortical map can be modelled by a log-polar transformation, see [9, 10]. This observation is important for understanding of the internal mechanism of image processing by the striate cortex in the contour completion problem.

This work continues our previous research, initiated in [11, 12], where we developed a computational framework for tracking of lines in images via data-driven sub-Riemannian geodesics in the Lie groups SE(2) and SO(3).



(a) Retina (b) Primary visual cortex

Figure 2: Experiment of Totelli: retina with spoke-wheel stimulus is retinotopically mapped onto stimulus in the striate cortex

The key idea in [**11**, **12**] was to include an external cost factor in the sub-Riemannian metric. The external cost was used for adaptation to image date. In this paper, we develop the idea of data-driven geodesics, but now in new context of modelling of visual system. Here, the external cost is added for adaptation to nonuniform distribution of photoreceptors. Based on the results of Florack [**10**], we propose a natural construction of the external cost for a sub-Riemannian metric in SO(3).

The paper has the following structure. It starts from introduction, where we retrospect history of the problem. Then, in Section 1, we prepare the necessary mathematical background for description of our model. To this end we state a problem $\mathbf{P_{curve}}$ on a sphere, and lift it to a sub-Riemannian problem $\mathbf{P_{mec}}$ in SO(3). Next, we analyse the problem $\mathbf{P_{mec}}$ and derive the Hamiltonian system of Pontryagin Maximum Principle, that describes the sub-Riemannian geodesics. Afterwards, in Section 2, we formulate our model for contour completion on the retinal hemisphere and illustrate it by a simulation of association field lines. Finally, we summarize the work in the conclusion.

History of the problem

An important discovery of neurophysiology of vision of mammals was done by Hubel and Wiesel in 1959 (Nobel prize in 1981), who showed that in striate cortex of a cat, there exist groups of neurons sensitive to positions and directions. Here, in the first stage of processing, the image is lifted by the brain to the extended space of positions and directions. In [2] a sub-Riemannian structure on the Heisenberg group was proposed for contour perception and completion.

This was refined subsequently in [3] as a sub-Riemannian problem on the SE(2) group. This means that one considers the following optimization of curves $\gamma(t) = (x(t), y(t), \theta(t))$ in SE(2):

$$\int_0^T \sqrt{\xi^2 \left(\dot{x}(t)^2 + \dot{y}(t)^2\right) + \dot{\theta}(t)^2}\, \mathrm{d}t \to \min,$$
$$\text{under constraint } \theta(t) = \arg(\pm\dot{x}(t) \pm \mathrm{i}\,\dot{y}(t)),$$

with a free time $T > 0$, that we can by reparameterization invariance set $T = 1$. This constraint is equivalent to imposing $\langle -\sin\theta(t)\,\mathrm{d}x + \cos\theta(t)\,\mathrm{d}y, \dot{\gamma}(t)\rangle = 0$.

Such curves may exhibit cusps on their projection to the image plane [13]. The set of terminal conditions for which cusps do not occur are determined in [14], where association field lines [15] in the psychology of vision are modeled by cuspless sub-Riemannian geodesics. Such curves were shown to be global minimizers, which followed by the optimal synthesis [16, 17] on $(\mathrm{SE}(2), \Delta, \mathcal{G}_\xi)$, with the distribution

$$\Delta = ker(-\sin\theta\,\mathrm{d}x + \cos\theta\,\mathrm{d}y)$$

and with the metric tensor

$$\mathcal{G}_\xi = \xi^2\,(\cos\theta\,\mathrm{d}x + \sin\theta\,\mathrm{d}y) \otimes (\cos\theta\,\mathrm{d}x + \sin\theta\,\mathrm{d}y) + \mathrm{d}\theta \otimes \mathrm{d}\theta.$$

For a semidiscrete version of this model see [18]. Furthermore, a possible extension of the model was proposed in [4], where the spherical nature of the retina is included. However, this is only done for the case of the metric, left-invariant w.r.t. action of SO(3) and right-invariant w.r.t. action of one-parametric subgroup SO(2). In that case, the optimal synthesis was obtained. Note, that the same result was independently obtained by Berestovskii and Zubareva [19], using a different technique. Also note, that the general case $\xi > 0$, in contrast to SE(2) case [14, 17], does not follow by a simple scaling homothety. A projective version of the problem for the case $\xi = 1$ was studied in [20].

In Section 2, we refine the model of Boscain and Rossi [4] by taking into account the nonuniform distribution of photoreceptors on the retina, and by considering the general case of sub-Riemannian metric in SO(3), where we control the stiffness of curves via the parameter $\xi > 0$.

1. Problem $\mathbf{P_{curve}}$ on a Sphere and Problem $\mathbf{P_{mec}}$ in SO(3)

In this section we provide mathematical formalism to our spherical model of V1. We start from mathematical notations used in this work. Then, we formulate a variational problem $\mathbf{P_{curve}}$ on a sphere, that will be our point of departure. We lift this problem to a sub-Riemannian problem $\mathbf{P_{mec}}$ in SO(3). Here we rely on our previous work [**12**], where we have shown that solution to the problem $\mathbf{P_{curve}}$ is obtained by projection of certain sub-Riemannian geodesics in SO(3) to the sphere S^2. Afterwards, we apply the Pontryagin Maximum Principle and derive the Hamiltonian system that describes the geodesics.

1.1. Mathematical foundation and notations

The Lie group SO(3) is the group of all rotations about the origin in $\mathbb{R}^3$. We shall denote a counter-clockwise rotation around axis $\mathbf{a} \in S^2$ with angle ϕ via $R_{\mathbf{a},\phi}$. In particular, for rotations around standard axes

$$\mathbf{e}_1 = (1,0,0)^T, \quad \mathbf{e}_2 = (0,1,0)^T, \quad \mathbf{e}_3 = (0,0,1)^T.$$

We use representation of SO(3) by 3×3 matrices

$$R(x,y,\theta) = R_{\mathbf{e}_3,y} R_{\mathbf{e}_2,-x} R_{\mathbf{e}_1,\theta} =$$

$$\text{(1)} \qquad \begin{pmatrix} cx\,cy & -sx\,cy\,s\theta - sy\,c\theta & sy\,s\theta - sx\,cy\,c\theta \\ cx\,sy & cy\,s\theta - sx\,sy\,s\theta & -cy\,s\theta - sx\,sy\,c\theta \\ sx & cx\,s\theta & cx\,c\theta \end{pmatrix},$$

where $cx = \cos x$, $cy = \cos y$, $c\theta = \cos\theta$, $sx = \sin x$, e.t.c., and

$$(x,y,\theta) \in \mathbb{R}/\{2\pi\mathbb{Z}\} \times \mathbb{R}/\{2\pi\mathbb{Z}\} \times \mathbb{R}/\{2\pi\mathbb{Z}\}.$$

The Lie group SO(3) defines an associated Lie algebra

$$\mathrm{so}(3) = T_{\mathrm{Id}}(\mathrm{SO}(3)) = \mathrm{span}(A_1, A_2, A_3),$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $T_{\mathrm{Id}}(\mathrm{SO}(3))$ denotes the tangent space at the unity element.

The non-zero Lie brackets between A_i are given by

$$\text{(2)} \qquad [A_1, A_2] = A_3, \quad [A_1, A_3] = -A_2, \quad [A_2, A_3] = A_1.$$

There is a natural isomorphism between so(3) and Lie algebra $\mathcal{L}(\mathrm{SO}(3))$ of left-invariant vector fields on SO(3), where commutators of vector fields in $\mathcal{L}(\mathrm{SO}(3))$ correspond to the matrix commutators in so(3)

$$[RA, RB] = R[A, B], \quad A, B \in \mathrm{so}(3),\ R \in \mathrm{SO}(3)\,.$$

In matrix form, we define $\mathcal{L}(\mathrm{SO}(3)) = \mathrm{span}(\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3)$, where

$$\begin{cases} \mathrm{X}_1(x, y, \theta) = -R(x, y, \theta)A_2, \\ \mathrm{X}_2(x, y, \theta) = R(x, y, \theta)A_1, \\ \mathrm{X}_3(x, y, \theta) = R(x, y, \theta)A_3, \end{cases} \tag{3}$$

We call a spherical projection the following map (see Fig. 3):

$$\mathrm{SO}(3) \ni R \mapsto R\,\mathbf{e}_1 \in S^2. \tag{4}$$

In coordinates (x, y, θ), defined by (1), we have

$$R(x, y, \theta)\,\mathbf{e}_1 = \begin{pmatrix} \cos x \cos y \\ \cos x \sin y \\ \sin x \end{pmatrix} = \mathbf{n}(x, y) \in S^2.$$

So we see, that (x, y) are spherical coordinates on S^2.

Remark 1.1. *Note, that due to physical construction of an eye it is enough to consider the problem $\mathbf{P_{curve}}$ on a hemisphere. This allows us to consider only the domain $(x, y) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$. From mathematical point of view, restriction to a hemisphere releases from a problem with multiple charts for covering full S^2.*

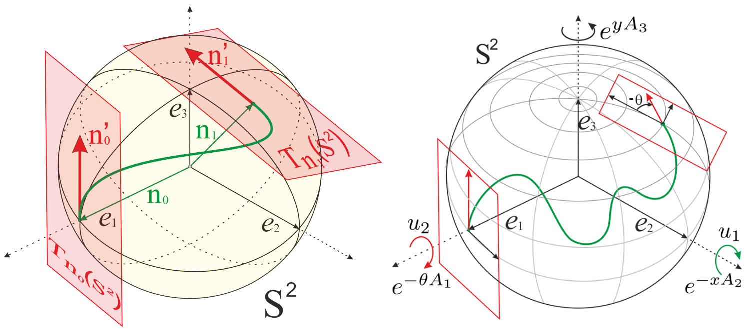
1.2. Problem $\mathbf{P_{curve}}$ on S^2 and $\mathbf{P_{mec}}$ in SO(3)

Let $S^2 = \{\mathbf{n} \in \mathbb{R}^3 \mid \|\mathbf{n}\| = 1\}$ be a sphere of unit radius. We consider the problem $\mathbf{P_{curve}}$ (see Fig. 3), which is for given boundary points $\mathbf{n}_0, \mathbf{n}_1 \in S^2$ and directions $\mathbf{n}_0' \in T_{\mathbf{n}_0}(S^2)$, $\mathbf{n}_1' \in T_{\mathbf{n}_1}(S^2)$, $\|\mathbf{n}_0'\| = \|\mathbf{n}_1'\| = 1$ to find a smooth curve $\mathbf{n}(\cdot) : [0, l] \to S^2$, that satisfies the conditions

$$\mathbf{n}(0) = \mathbf{n}_0,\ \mathbf{n}(l) = \mathbf{n}_1, \quad \mathbf{n}'(0) = \mathbf{n}_0',\ \mathbf{n}'(l) = \mathbf{n}_1', \tag{5}$$

and, for given $\xi > 0$, minimizes the functional

$$\mathcal{L}(\mathbf{n}(\cdot)) := \int_0^l \mathfrak{C}(\mathbf{n}(s))\sqrt{\xi^2 + k_g^2(s)}\,\mathrm{d}s, \tag{6}$$



(a) Problem $\mathbf{P_{curve}}$ (b) Problem $\mathbf{P_{mec}}$

Figure 3: Problem $\mathbf{P_{curve}}$ on S^2 of finding a curve with given boundary conditions (5) that minimizes (6), is lifted to the problem $\mathbf{P_{mec}}$ in SO(3), parameterized by angles x, y, θ of rotation about basis axes, see (1.1)

where $k_g(s)$ denotes the geodesic curvature of $\mathbf{n}(\cdot)$ evaluated at time s, and $\mathfrak{C} : S^2 \to [\delta, +\infty)$, $\delta > 0$, is a smooth function "external cost".

The problem was studied in our previous work [**12**], where we have shown that minimizers of $\mathbf{P_{curve}}$ are given by spherical projection of certain minimizers in a sub-Riemannian problem $\mathbf{P_{mec}}$ in SO(3).

1.3. Sub-Riemannian Problem $\mathbf{P_{mec}}$ in SO(3)

The problem $\mathbf{P_{mec}}$ is given by the following optimal control problem:

$$\dot{\gamma}(t) = \textstyle\sum_{i=1}^{2} u_i(t)\, X_i|_{\gamma(t)}, \qquad \text{for } t \in [0, T],$$
$$\gamma(0) = Id, \quad \gamma(T) = g_1, \qquad \gamma(t) \in \mathrm{SO}(3), \quad (u_1(t), u_2(t)) \in \mathbb{R}^2,$$
$$l(\gamma(\cdot)) = \textstyle\int_0^T \mathcal{C}(\gamma(t))\sqrt{\xi^2 u_1^2(t) + u_2^2(t)}\, dt \to \min .$$

Here, the terminal time T is free; X_i are basis left-invariant vector fields in SO(3), recall (3); and $\mathcal{C} : \mathrm{SO}(3) \to [\delta, +\infty)$, $\delta > 0$ is an external cost.

We parameterize $\gamma(\cdot)$ by the SR-arclength, that is defined by

$$\mathcal{C}(\gamma(t))\sqrt{\xi^2 u_1^2(t) + u_2^2(t)} = 1.$$

The Cauchy-Schwartz inequality implies that the minimization problem for the sub-Riemannian length functional l is equivalent to the minimization problem for the action functional

$$J = \frac{1}{2}\int_0^T \mathcal{C}^2(\gamma(t))(\xi^2 u_1^2(t) + u_2^2(t))\, dt,$$

with fixed T.

Next, we apply Pontryagin Maximum Principle in invariant formulation [21]. The control dependent Hamiltonian reads as

$$H_u(\lambda, g) = u_1 h_1 + u_2 h_2 - \frac{1}{2}\mathcal{C}^2(g)\left(\xi^2 u_1^2 + u_2^2\right),$$

where $g \in \mathrm{SO}(3)$, $h_i = \langle \lambda, X_i \rangle$, $\lambda \in T_g^*(\mathrm{SO}(3))$ are basis left-invariant Hamiltonians, linear on fibers of the cotangent bundle $T^*(\mathrm{SO}(3))$.

The (maximized) Hamiltonian

$$H = \max_{(u_1,u_2)\in\mathbb{R}^2} H_u(\lambda, g) = \frac{1}{2\,\mathcal{C}(g)^2}\left(\frac{h_1^2}{\xi^2} + h_2^2\right)$$

follows from the expression for the extremal controls

$$\frac{dH_u}{du_i} = 0,\ i = 1, 2 \quad \Rightarrow \quad u_1 = \frac{h_1}{\xi^2 \mathcal{C}^2}, \quad u_2 = \frac{h_2}{\mathcal{C}^2}.$$

Now, we derive the Hamiltonian system of PMP. The vertical part (for momentum components h_i) is given by

$$\dot{h}_i = \{H, h_i\},$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket (see [21]).

Using the standard relation between Poisson and Lie brackets $\{h_i, h_j\} = \langle \lambda, [X_i, X_j] \rangle$, recall (2), we get,

$$\begin{aligned}
\{H, h_1\} &= \frac{X_1(\mathcal{C})}{\mathcal{C}} - \frac{h_2 h_3}{\mathcal{C}^2}, \\
\{H, h_2\} &= \frac{X_2(\mathcal{C})}{\mathcal{C}} + \frac{1}{\xi^2}\,\frac{h_1 h_3}{\mathcal{C}^2}, \\
\{H, h_3\} &= \frac{X_3(\mathcal{C})}{\mathcal{C}} + \left(1 - \frac{1}{\xi^2}\right)\frac{h_1 h_2}{\mathcal{C}^2},
\end{aligned}$$

where $X_i(\mathcal{C})$ denotes the derivative of the function $\mathcal{C}$ along X_i.

The horizontal part of the Hamiltonian system is obtained by substitution of extremal controls in the control system

$$\dot{\gamma}(t) = u_1(t) X_1|_{\gamma(t)} + u_2(t) X_2|_{\gamma(t)}. \tag{7}$$

A requested sub-Riemannian geodesic γ is obtained by integration of (7).

Switching to the coordinate chart (x, y, θ), recall Section 1.1, and restriction to the hemisphere $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$, recall Remark 1.1, provides the following result.

THEOREM 1.1. *In problem* $\mathbf{P_{mec}}$, *any sub-Riemannian geodesic* $\gamma(t) = (x(t), y(t), \theta(t))$, *whose spherical projection (4) belongs to the hemisphere, is obtained by integration of the Hamiltonian system*

$$(8)\qquad \begin{cases} \dot{h}_1 = \frac{X_1(\mathcal{C})}{\mathcal{C}} - \frac{h_2 h_3}{\mathcal{C}^2}, \\ \dot{h}_2 = \frac{X_2(\mathcal{C})}{\mathcal{C}} + \frac{1}{\xi^2}\,\frac{h_1 h_3}{\mathcal{C}^2}, \\ \dot{h}_3 = \frac{X_3(\mathcal{C})}{\mathcal{C}} + \left(1 - \frac{1}{\xi^2}\right)\frac{h_1 h_2}{\mathcal{C}^2} \end{cases} \qquad \begin{cases} \dot{x} = \frac{h_1}{\xi^2\mathcal{C}^2}\cos\theta, \\ \dot{y} = -\frac{h_1}{\xi^2\mathcal{C}^2}\sec x \sin\theta, \\ \dot{\theta} = \frac{h_1}{\xi^2\mathcal{C}^2}\sin\theta\tan x + \frac{h_2}{\mathcal{C}^2} \end{cases}$$

— *vertical part,* — *horizontal part,*

with the initial conditions

$$(9)\qquad h_1(0) = h_1^0,\ h_2(0) = h_2^0,\ h_3(0) = h_3^0,\ x(0) = y(0) = \theta(0) = 0,$$

for $t \in [0, T]$, *where* $T > 0$ *is sufficiently small.*

2. Model of contour completion on the retinal sphere

Now we have all the background to formulate a model of contour completion on the retinal hemisphere.

An appropriate external cost $\mathfrak{C}$ on the hemisphere encodes the nonuniform distribution of photoreceptors. To formulate the associated model for contour completion, we lift $\mathfrak{C}(x, y)$ from S^2 to the Lie group SO(3). In our model, this lift is simply done by $C(x, y, \theta) = \mathfrak{C}(x, y)$, for all $\theta \in S^1$.

Then, our model for contour completion on the retinal hemisphere can be written as the minimization of

$$\int_0^T C(\gamma(t))\sqrt{\xi^2\big(\dot{x}(t)^2 + \dot{y}(t)^2\cos^2 x(t)\big) + \big(\dot{\theta}(t) + \dot{y}(t)\sin x(t)\big)^2}\,\mathrm{d}t,$$

among the curves $\gamma(t) = (x(t), y(t), \theta(t))$ in SO(3) satisfying the horizontality constraint $\theta(t) = -\arg(\pm\dot{x}(t) \pm \mathrm{i}\,\dot{y}(t)\cos x(t))$ and the boundary conditions

$$x(0) = y(0) = \theta(0) = 0,\ x(T) = x_1,\ y(T) = y_1,\ \theta(T) = \theta_1,$$

with free $T > 0$.

The parameter ξ encodes the balance between the "spatial" displacement on the sphere S^2 and "angular" displacement in θ direction. By considering the general case $\xi > 0$, we obtain a general case of sub-Riemannian metric in SO(3) (see [**22**]). Note, that in [**4**], [**19**], only the symmetric case $\xi = 1$ was studied.

Remark 2.1. *According to our model, the lift of an image by the cortex can be interpreted as a map from S^2 (retina) to S^3 (cortex), which is double cover of* SO(3)*. Then, the Hopf fibration* [**23**] *naturally appears as a fiber bundle, where the base is S^2 and a fiber is S^1, see Fig. 4.*

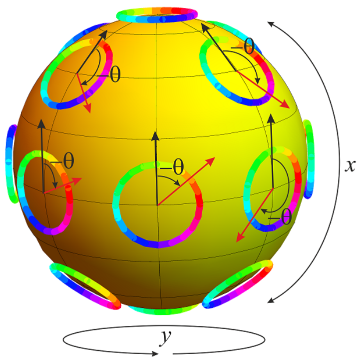


Figure 4: Hopf fibration — circle bundle with the base S^2

Now, we formulate a criteria of good continuation.

Criteria of good continuation. *Given a boundary condition $g_1 = (x_1, y_1, \theta_1)$ is perceptually connected to $e = (0, 0, 0)$, if the sub-Riemannian distance $d(e, g_1) \leq T$ for some fixed $T > 0$. Here, we also assume that g_1 is chosen such that it can be connected to e with a minimizing geodesic whose spherical projection does not have a cusp* [**12**, **14**].

2.1. Construction of the external cost

Now, as the distribution of photoreceptors is nonuniform (see Fig. 1) on the retina, this can be included in our model by putting an appropriate external cost $\mathfrak{C}$ on a hemisphere. In this section, we develop this idea.

The nonuniform distribution of photoreceptors could provide a natural reason for cortical magnification [**9**, **10**]. Cortical magnification describes how many neurons in an area of the visual cortex are responsible for processing a stimulus of a given size on the retina.

A mathematical description of phenomenon of cortical magnification is proposed by Florack in [**9**]. There, the slightly modified log-polar coordinates on the geometrical retina were introduced. According to the model of Florack, the log-polar coordinates are rectified in the primary visual cortex. Thus cortical magnification is naturally modelled by a log-polar transform. Such a model is supported by experimental results of Tootell et al. [8], who studied functional anatomy of macaque striate cortex, see Fig. 2.

From the mathematical point of view, it is well-known that the log-polar coordinates have a cumbersome and unrealistic singularity at the origin. In reality, this singularity never appears in the biological system since even at the centre of the foveola there are still finite number of photoreceptors. The problem of the singularity was resolved in the consequent work of Florack [10] by a slight modification of the log polar coordinates (cf. Eq. (10) below).

We base our construction of the external cost on the model [10]. The external cost puts the penalty in the distance measure when moving outside the foveola (central point of the macula).

To illustrate the phenomena of cortical magnification, let us consider the map from the canonical coordinates in the cortex to the retina (see [10, eq. (20)]). This map provides a natural coordinate chart on the retinal hemisphere, given by

$$\begin{cases} x = \sinh(p)\cos\phi, \\ y = \sinh(p)\sin\phi, \end{cases} \tag{10}$$

where $\phi = \arg(x + \mathrm{i}\, y) \in S^1$ is a polar angle, and $p \geq 0$ is the canonical coordinate, that encodes the logarithm of the distance between the fovea and a given point (x, y), introduced by [10, eq. (20)].

According to [10, eq. (13)], the integrated retino-cortical magnification is given by

$$v(x, y) = \frac{\log(1 + (x^2 + y^2)/a^2)}{\log(1 + T_0^2)}, \tag{11}$$

where $a > 0$ is a parameter, that represents a transient radius separating the geometric foveola, and T_0 is a constant, that encodes the maximal size of retinal region involved in visual perception. The phenomenological value $T_0 \approx 95$ and $a \approx \frac{R}{10}$ were estimated in [10]. Here R is physical size of a radius of the eye. In our model, we represent the eye by a unit sphere S^2 and use the eyeball radius to express lengths. Thus, we set $a = 0.1$.

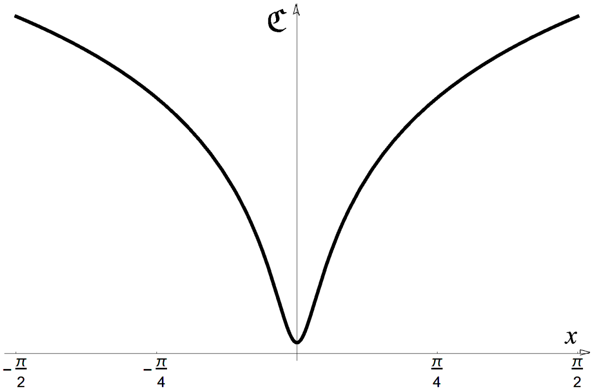


Figure 5: The graph of $\mathfrak{C}(x, 0)$ given by (12)

We construct the external cost $\mathfrak{C}(x, y)$ so, that it does not put a penalty in the foveola $\mathfrak{C}(0, 0) = 1$, and it penalizes a motion outside the foveola proportionally to the cortical magnification factor (11)

$$\mathfrak{C}(x, y) = 1 + v(x, y), \tag{12}$$

See the profile of the external cost (12) in Fig. 5.

3. Simulation of the Association Field Model

Gestalt laws have been proposed for several phenomena of visual perception. Among them the law of good continuation plays a central role for perceptual completion. The same principle of good continuation has been also found in psychophysical experiments of Field, Hayes and Hess [**15**]. Those experiments have resulted to the notion of association field, which describes the set of possible subjective contour starting from a point with a horizontal orientation (see Fig. 6, right).

Here, we provide a simple experiment for a simulation of the association field lines on the retinal hemisphere via spherical projections of the sub-Riemannian length minimizers of the problem $\mathbf{P_{mec}}$, where we rely on Theorem 1.1 and employ the fact that sufficiently short arcs of geodesics are minimizers [**21**].

See the result in Fig. 6. To perform this experiment we set $\xi = 2$, $h_3(0) = 0$, $h_2(0) \in \{-0.7, -0.525, -0.35, -0.175, 0, 0.175, 0.35, 0.525, 0.7\}$ and $T = \frac{3}{4}\pi$. Here we note that in order to get an association field that is symmetric around the x-axis, one must set $h_3(0) = 0$ and let $h_2(0)$ run in a symmetric interval. This follows by the Hamiltonian system (8).

We can observe, that the association field on the retinal sphere via projection of the sub-Riemannian geodesics in SO(3) indeed leads to a close approximation of the association field from the original work of Field, Hayes and Hess [15].

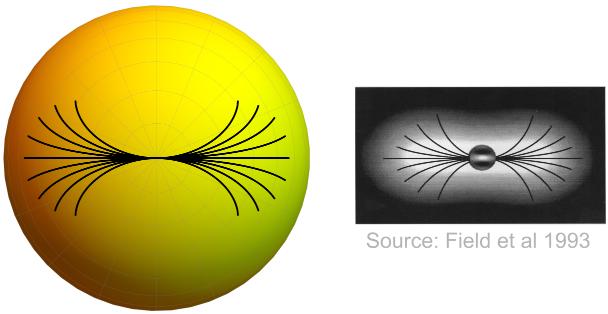


Figure 6: The association field obtained by the experiment from Section 3 versus the original result of Field, Hayes and Hess

Conclusion and discussion

In this work, we have proposed the cortical based model for contour completion on the retinal sphere, which includes both a stiffness parameter $\xi > 0$ and accounts for the nonuniform distributions of photoreceptors on the retinal hemisphere via an external non-uniform cost. This realistic external cost (12) relies on mathematical model of cortical magnification [10]. In this article we have provided a novel mathematical formulation of good continuation. Applying this principle (together with a necessary symmetry constraint on initial momenta) indeed leads to a close approximation of the association field [15] by Field, Hayes and Hess, as shown in Fig. 6.

For future work, we plan to apply the proposed model of contour completion for reconstruction of partially corrupted contours in real images. There we plan to employ our criteria of good continuation and to develop the software that selects only the sufficiently close boundary conditions (in sense of sub-Riemannian distance) among all the boundary conditions, induced by the image locations, where the contours are corrupted. The selected boundary conditions will be connected via data-driven sub-Riemannian length minimizers, computed via our numerical approach [**11**, **12**]. Furthermore, automatic training of the parameters ξ and T is an interesting topic for further investigation.

Acknowledgments

The authors thank G.R. Sanguinetti and E.J. Bekkers for many helpful comments and suggestions on this work. We thank Prof. G. Citti for pointing us to the idea of including the nonuniform distribution of photoreceptors on the retina.

References

[1] D.H. Hubel, T.N. Wiesel. "Receptive fiedls, binocular interaction and functional architecture in the cat's visual cortex", *The Journal of Physiology*, **160**:1 (1962), pp. 106–154. ↑ 231

[2] J. Petitot. "The neurogeometry of pinwheels as a sub-Riemannian contact structure", *Journal of Physiology-Paris*, **97**:2-3 (2003), pp. 265–309. ↑ 232,234

[3] G. Citti, A. Sarti. "A cortical based model of perceptual completion in the roto-translation space", *Journal of Mathematical Imaging and Vision*, **24**:3 (2006), pp. 307–326. ↑ 232,234

[4] U. Boscain, F. Rossi. "Invariant Carnot-Caratheodory metrics on S^3, $SO(3)$, $SL(2)$ and Lens Spaces", *SIAM, Journal on Control and Optimization*, **47** (2008), pp. 1851–1878. ↑ 232,234,239

[5] G. Osterberg. "Topography of the Layer of Rods and Cones in the Human Retina", *Acta Ophthalmologica (Supplementum)*, **6** (1935), pp. 1–103. ↑ 232

[6] B. M. ter Haar Romeny, *Front-End Vision and Multi-Scale Image Analysis*, Computational Imaging and Vision, vol. **27**, Springer Netherlands, 2003, 484 p. ↑ 232

[7] P.M. Daniel, D. Whiteridge. "The representation of the visual field on the cerebral cortex in monkeys", *The Journal of Physiology*, **159**:2 (1961), pp. 203–221. ↑ 232

[8] R. B. H. Tootell, E. Switkes, M. S. Silverman, S. J. Hamilton. "Functional anatomy of macaque striate cortex. II. Retinotopic organization", *Journal of Neuroscience*, **8**:5 (1988), pp. 1531–1568. ↑ 232,241

[9] L. Florack. "A Geometric Model for Cortical Magnification", *Biologically Motivated Computer Vision*, Lecture Notes in Computer Science, vol. **1811**, eds. S.-W. Lee, H. H. Bulthoff, T. Poggio, 2000, pp. 574–583. ↑ 232,240

[10] L. Florack. "Canonical Coordinates for Retino-Cortical Magnification", *Axioms*, **3**:1 (2014), pp. 70–81. ↑ 232,233,240,241,243

[11] E.J. Bekkers, R. Duits, A. Mashtakov, G.R. Sanguinetti (joint main authors). "A PDE Approach to Data-driven Sub-Riemannian Geodesics in SE(2)", *SIAM Journal on Imaging Sciences*, **8**:4 (2015), pp. 2740–2770. ↑ 232,233,244

[12] A. Mashtakov, R. Duits, Yu. Sachkov, E. Bekkers, I. Beschastnyi. "Tracking of Lines in Spherical Images via Sub-Riemannian Geodesics on SO(3)", *Journal of Mathematical Imaging and Vision*, 2016 (to appear), URL: https://arxiv.org/abs/1604.03800 ↑ 232,233,235,237,240,244

[13] U. Boscain, R. Duits, F. Rossi, Y. L. Sachkov. "Curve Cuspless Reconstruction via sub-Riemannian Geometry", *ESAIM: Control, Optimisation and Calculus of Variations*, **20**:3 (2014), pp. 748–770. ↑ 234

[14] R. Duits, U. Boscain, F. Rossi, Y.L. Sachkov. "Association Fields via Cuspless Sub-Riemannian Geodesics in SE(2)", *Journal of Mathematical Imaging and Vision*, **49**:2 (2014), pp. 384–417. ↑ 234,240

[15] D.J. Field, A. Hayes, R. Hess. "Contour integration by the human visual system: Evidence for a local "association field"", *Vision Research*, **33**:2 (1993), pp. 173–193. ↑ 234,242,243

[16] I. Moiseev, Yu.L. Sachkov. "Maxwell strata in sub-Riemannian problem on the group of motions of a plane", *ESAIM: Control, Optimisation and Calculus of Variations*, **16**:2 (2010), pp. 380–399. ↑ 234

[17] Yu.L. Sachkov. "Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane", *ESAIM: Control, Optimisation and Calculus of Variations*, **17**:2 (2011), pp. 293–321. ↑ 234

[18] U. Boscain, J.-P. Gauthier, R. Chertovskih, A. Remizov. "Hypoelliptic diffusion and human vision: a semidiscrete new twist", *SIAM J. Imaging Sci.*, **7**:2 (2014), pp. 669–695. ↑ 234

[19] V. Berestovskii, I. Zubareva. "Sub-Riemannian distance in the Lie groups SU(2) and SO(3)", *Siberian Advances in Mathematics*, **26**:2 (2016), pp. 77–89. ↑ 234,239

[20] U. Boscain, F. Rossi. "Projective Reed-Shepp Car on S^2", *ESAIM: Control, Optimisation and Calculus of Variations*, **16**:2 (2010), pp. 275–297. ↑ 234

[21] A. A. Agrachev, Yu. L. Sachkov, *Control Theory from the Geometric Viewpoint*, Encyclopaedia of Mathematical Sciences, vol. **87**, Springer-Verlag, Berlin–Heidelberg, 2004, 412 p. ↑ 238,242

[22] A. Agrachev, D. Barilari. "Sub-Riemannian structures on 3D lie groups", *Journal of Dynamical and Control Systems*, **18**:1 (2012), pp. 21–44. ↑ 239

[23] D.-C. Chang, I. Markina, A. Vasil'ev. "Sub-Riemannian geometry on the 3-D sphere", *Complex analysis and operator theory*, **3**:2 (2009), pp. 361–377. ↑ 240

Sample citation of this publication:

A. Mashtakov, R. Duits. "A Cortical Based Model for Contour Completion on the Retinal Sphere", *Program systems: Theory and applications*, 2016, **7**:4(31), pp. 231–247. URL: http://psta.psiras.ru/read/psta2016_4_231-247.pdf

About the authors:

Alexey Mashtakov

A. Mashtakov received his M.S. in Applied Mathematics and Computer Science at University of Pereslavl in 2009. He received his Ph.D. in Applied Mathematics and Computer Science in PSI RAS in 2013. He was a post-doc at Department of Biomedical Engineering at the TU/e from 2014 till 2016. Currently he is a senior researcher of Control Processes Research Centre at Program Systems Institute, Pereslavl- Zalessky (Russian Academy of Sciences). His research interests are geometric control theory, optimal control, control theory on Lie Groups, nonholonomic systems, sub-Riemannian geometry and their applications in image processing, robotics and modelling of vision.

e-mail: alexey.mashtakov@gmail.com

Remco Duits

R. Duits received his M.Sc. degree (cum laude) in Mathematics in 2001 at the TU/e, The Netherlands. He received his Ph.D. degree (cum laude) at the Department of Biomedical Engineering at the TU/e. Currently he is an associate professor at the Department of Applied Mathematics and Computer Science and affiliated at the Biomedical Image Analysis group at the Biomedical Engineering Department. He has received an ERC-StG 2013 personal grant (Lie analysis, no. 335555). His research interests include group theory, functional analysis, differential geometry, PDE's, harmonic analysis, geometric control, and their applications to biomedical imaging and vision.

e-mail: R.Duits@tue.nl

УДК 517.977

А. П. Маштаков, Р. Дайтс. *Сферическая модель первичной зрительной коры головного мозга человека.*

Аннотация. В работе предложено и исследовано естественное сферическое обобщение модели Petitot–Citti–Sarti первичной зрительной коры головного мозга человека. Уточнение осуществляется путем включения кривизны сетчатки. В предлагаемой модели сетчатка имеет форму полусферы. Это дает лучшее приближение, чем аппроксимация сетчатки плоскостью (использованная в модели Petitot–Citti–Sarti). Возникающая при этом задача поиска кривых, минимизирующих компромисс между длиной и геодезической кривизной кривой на поверхности сферы (вариационный принцип, в соответствии с которым человеческий мозг восстанавливает скрытые от наблюдения контуры), с заданными граничными точками и направлениями на поверхности сферы, решается путем подъема задачи на группу Ли SO(3). Неоднородность распределения светочувствительных рецепторов на сетчатке глаза также учтена, путем включения внешней стоимости в субриманову структуру. (*In English*).

Ключевые слова и фразы: поле ассоциаций, зрительная кора, субримановы геодезические, группа Ли.

Пример ссылки на эту публикацию:

А. П. Маштаков, Р. Дайтс. «Сферическая модель первичной зрительной коры головного мозга человека», *Программные системы: теория и приложения*, 2016, **7**:4(31), с. 231–247. (*Англ*).

URL: http://psta.psiras.ru/read/psta2016_4_231-247.pdf

Работа выполнена при поддержке РФФИ (грант 16-31-60083_мол_а_дк)[1].
Работа выполнена при поддержке FP7/2007-2013/ERC (грант № 335555)[2].