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Approximate optimal control synthesis for nonuniform discrete systems with linear-quadratic state

Abstract. Nonuniform discrete systems linear-quadratic over its state are the subject of intense study in optimal control theory. This work presents an approximate optimal control synthesis method in this class based on Krotov's sufficient optimality conditions and illustrates it with a simple example.

Key words and phrases: nonuniform discrete systems, sufficient optimality conditions, approximate optimal control synthesis.

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Introduction

Nonuniform control systems are the subject of intense study in optimal control theory. Their state in terms of controlled differential and discrete systems depends on time. Approaches to the development of their mathematical models and investigations, as well as the terminology, e.g., systems with variable structure [?1], discrete-continuous [?2], logic-dynamic [?3, ?4], impulsive [?5], and hybrid systems [?6, ?7], are very diverse. One possible approach to study optimal control problems for such systems is the generalization of Krotov's sufficient optimality conditions for them [?8, ?9]. In [?2, ?10, ?11, ?12], the authors propose a two-level mathematical model of a discrete-continuous system (DCS). The lower level describes the continuous controllable processes at the individual stages. The upper level integrates system descriptions into a unique process and controls the functioning of the entire system as a whole to ensure the minimum of the functional. The work [?12] establish sufficient optimality conditions for the model and presents control improvement methods.

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Work [?13] extends this model to the case where the lower level contains discrete systems. This case is called the *nonuniform discrete systems* (NDS). In [?14], the authors propose an iterative control improvement method for this model with the resolving Krotov functions linearly depend on the states at both levels. In the case of linear-quadratic nonuniform discrete systems, if we specify the Krotov functions as linear-quadratic, then we get the solution as an approximate synthesis of optimal control. The purpose of this work is to derive the method and illustrate it by example.

1. Nonuniform discrete systems with linear-quadratic state

We consider the NDS model representing a two-level controlled system of the form

$$x^0(k+1) = x^0(k) + \frac{1}{2}a(k)|x|^2 + b(k,u), \qquad a \geqslant 0,$$
$$x(k+1) = A(k)x(k) + B(k,u), \qquad x^0 \in \mathbb{R}, \qquad x \in \mathbb{R}^{m(k)},$$
$$k \in \mathbf{K} = \{k_I, k_I+1, ..., k_F\}, \qquad u \in \mathbf{U}(k,x) \subset \mathbb{R}^{r(k)},$$

where k is the step (stage) number, $\mathbf{U}(k,x)$ is the set given for each k and x, $A(k)$ is an $m(k)\times m(k)$ matrix, $B(k,u)$ is an $m(k)\times 1$ matrix, $a(k)$, $b(k,u)$ are given functions.

On some subset $\mathbf{K}' \subset \mathbf{K}$, $k_F \notin \mathbf{K}'$, there is a discrete system of the lower level

$$x^{d0}(k,t+1) = x^{d0}(k,t) + \frac{1}{2}a^d(k,t)|x^d|^2 + b^d(k,t,u^d), \qquad x^{d0} \in \mathbb{R},$$
$$x^d(k,t+1) = A^d(k,t)x^d + B^d(k,t,u^d), \qquad x^d \in \mathbf{X}^d(z,t) \subset \mathbb{R}^{n(k)},$$
$$t \in \mathbf{T}(z) = \{t_I(z), t_I(z)+1, \dots, t_F(z)\}, \qquad a^d \geqslant 0,$$
$$u^d \in \mathbf{U}^d\left(z,t,x^d\right) \subset \mathbb{R}^{p(k)},$$

where $A^d(k,t)$ is an $n(k)\times n(k)$ matrix, $B^d(k,t,u^d)$ is an $n(k)\times 1$ matrix, $a^d(k,t)$, $b^d(k,t,u^d)$ are given functions, the right-hand side operator is given by

$$x^0(k+1) = x^0(k) + \frac{1}{2}\beta(k)|x_F^d|^2, \qquad x(k+1) = x(k) + \zeta(k)x_F^d,$$
$$x^d(k,t_I) = \xi(k)x, \qquad x^{0d}(k,t_I) = \frac{1}{2}\xi^1(k)|x|^2, \qquad k \in \mathbf{K}',$$

where ξ, ξ^1, ζ, β are matrices of the corresponding dimensions. Here $z = (k,x)$ is a set of the upper-level variables playing the role of lower-level parameters.

The solution of this two-level system is the set

$$m = \left(x^0(k), x(k), u(k)\right),$$

where for $k \in \mathbf{K}'$: $u(k) = \left(u^d(k), m^d(k)\right)$, $m^d(k) \in \mathbf{D}^d\left(z(k)\right)$ (called a *nonuniform discrete process with linear-quadratic state*), where $m^d(k)$ is a discrete process $(x^{d0}, x^d(k,t), u^d(k,t))$, $t \in \mathbf{T}(z(k))$, and $\mathbf{D}^d(z)$ is the set of admissible processes m^d satisfying with the specified discrete system. Let us denote the set of elements m satisfying all the above conditions by $\mathbf{D}$ and call it a set of admissible nonuniform discrete processes with linear-quadratic state.

We consider the problem of finding the minimum on $\mathbf{D}$ of the functional $I = l^{\mathrm{T}} x\left(k_F\right) + \frac{1}{2}\kappa |x(k_F)|^2 + d$, where l is a vector, κ is a matrix, d is a constant, for fixed $k_I = 0$, $k_F = K$, $x\left(k_I\right)$ and additional constraints $x(k) \in \mathbf{X}(k)$, $x^d \in \mathbf{X}^d\left(z,t\right)$, where $\mathbf{X}(k)$, $\mathbf{X}^d\left(z,t\right)$ are given sets.

2. Basic constructions and sufficient optimality conditions

The sufficient optimality conditions for this model are derived by analogy with Krotov's sufficient conditions for discrete systems [?9] as follows. The discrete chains from $\mathbf{D}$ and $\mathbf{D}^d$ are excluded and scalar functions (functionals) $\varphi\left(k,x\right)$, $\varphi^c\left(z,t,x^d\right)$ are introduced. Then we construct a generalized Lagrangian by analogy with the Krotov Lagrangian for discrete systems:

$$\begin{gathered}
L = G\left(x\left(k_F\right)\right) - \sum_{\mathbf{K}\setminus\mathbf{K}'\setminus k_F} R(k, x(k), u(k)) + \\
+ \sum_{\mathbf{K}'} \Big(G^d(z) - \sum_{\mathbf{T}(z)\setminus t_F} R^d(z,t,x^d(k,t),u^d(k,t))\Big), \\
G\left(x\right) = F\left(x\right) + \varphi\left(k_F, x\right) - \varphi\left(k_I, x\left(k_I\right)\right), \\
R\left(k,x,u\right) = \varphi\left(k+1, f\left(k,x,u\right)\right) - \varphi\left(k,x\right), \\
G^d\left(k,z,\gamma^d\right) = -\varphi\left(k+1, \theta\left(k,z,\gamma^d\right)\right) + \varphi\left(k, x\left(k\right)\right) + \\
+ \varphi^d\left(k,z,t_F,x^d_F\right) - \varphi^d\left(k,z,t_I,x^d_I\right), \\
R^d\left(k,z,t,x^d,u^d\right) = \varphi^d\left(k,z,t+1, f^d\left(k,z,t,x^d,u^d\right)\right) - \varphi^d(k,z,t,x^d), \\
\mu^d\left(k,z,t\right) = \sup\left\{R^d\left(k,z,t,x^d,u^d\right) : x^d \in \mathbf{X}^d(k,z,t),\right. \\
\left. u^d \in \mathbf{U}^d\left(k,z,t,x^d\right)\right\}, \\
l^d\left(k,z\right) = \inf\left\{G^d\left(k,z,\gamma^d\right) : \left(\gamma^d\right) \in \mathbf{\Gamma}^d(k,z), \quad x^d \in \mathbf{X}^d(k,z,t_F)\right\}. \\
\mu\left(k\right) = \begin{cases} \sup\{R\left(k,x,u\right) : x \in \mathbf{X}(k), u \in \mathbf{U}\left(k,x\right)\}, & t \in \mathbf{K}\setminus\mathbf{K}', \\ -\inf\{l^d\left(z\right) : x \in \mathbf{X}\left(k\right), \quad u^v \in \mathbf{U}^v\left(k,x\right)\}, & k \in \mathbf{K}', \end{cases}
\end{gathered}$$

$$l = \inf\{G\left(x\right):\ x \in \mathbf{\Gamma} \cap \mathbf{X}\left(k_F\right)\}.$$

Here $\varphi\left(k, x\right)$ is an arbitrary functional, $\varphi^d\left(k, z, t, x^d\right)$ is an arbitrary parametric family of functionals (with parameters $k,\ z$), and f, θ, f^d denote the right-hand sides of disctete systems at the lower and upper-levels on $\mathbf{K}\backslash\mathbf{K}',\ \mathbf{K}',\ \mathbf{T}$, respectively.

THEOREM 1. *For any element $m \in \mathbf{D}$ and any $\varphi,\ \varphi^d$ the estimate is*

$$I(m) - \inf_{\mathbf{D}} I \leq \Delta = I(m) - l.$$

Let there be two processes $m^{\mathrm{I}} \in \mathbf{D}$ and $m^{\mathrm{II}} \in \mathbf{E}$ and functionals φ and φ^d such that $L\left(m^{\mathrm{II}}\right) < L\left(m^{\mathrm{I}}\right) = I\left(m^{\mathrm{I}}\right)$, and $m^{\mathrm{II}} \in \mathbf{D}$.

Then $I(m^{\mathrm{II}}) < I(m^{\mathrm{I}})$.

THEOREM 2. *Consider a sequence of processes $\{m_s\} \subset \mathbf{D}$ and functionals $\varphi,\ \varphi^d$ such that*

(1) $R\left(k, x_s\left(k\right), u_s\left(k\right)\right) \to \mu\left(k\right), \quad k \in \mathbf{K};$

(2) $R^d\left(z_s, t, x_s^d\left(t\right), u_s^d\left(t\right)\right) - \mu^d\left(z_s, t\right) \to 0, \quad k \in \mathbf{K}', \quad t \in \mathbf{T}\left(z_s\right);$

(3) $G^d\left(z_s, \gamma_s^d\right) - l^d\left(z_s\right) \to 0, \quad k \in \mathbf{K}';$

(4) $G\left(x_s\left(t_F\right)\right) \to l.$

Then $\{m_s\}$ is a minimizing sequence for I on $\mathbf{D}$.

The proofs of these theorems are given in [?13].

3. Approximate optimal control synthesis and algorithm

To construct an approximate optimal control synthesis method, here we use Krotov's sufficient optimality conditions, the extension [?gurman] and localization priciples [?gurman_rasina]. Suppose that $\mathbf{U}(k, x) = \mathbb{R}^{r(k)}$, $\mathbf{U}^d\left(z, t, x^d\right) = \mathbb{R}^{p(k)}$ and the used constructions of sufficient optimality conditions are such that all the following operations are valid. The method for solving this problem is generated via some improvement operator $\eta(m) : \mathbf{D} \to \mathbf{D}$ such that $I(\eta(m)) \leq I(m)$ [?15].

We define an auxiliary functional [?gurman_rasina]

$$I_\alpha = L_\alpha = \alpha I + \frac{1}{2}(1 - \alpha)\left(\sum_{\mathbf{K}\backslash\mathbf{K}'\backslash k_F} |\Delta u\left(k\right)|^2 + \sum_{\mathbf{K}'} \sum_{\mathbf{T}(z)\backslash t_F} |\Delta u^d\left(k, t\right)|^2\right),$$

where $\alpha \in [0,1]$ and its increment:

$$\Delta L_\alpha \approx G_x^{\mathrm{T}} \Delta x - \sum_{\mathbf{K}\setminus\mathbf{K}'} \left(R_x^{\mathrm{T}} \Delta x + R_u^{\mathrm{T}} \Delta u + \frac{1}{2} \Delta u^{\mathrm{T}} R_{uu} \Delta u \right) -$$
$$- \sum_{\mathbf{K}'} \Big((G_x^{d\mathrm{T}} \Delta x + G_{x_F^d}^{d\mathrm{T}} \Delta x_F^d) -$$
$$- \sum_{\mathbf{T}(z)\setminus t_F} (R_x^{d\mathrm{T}} \Delta x + R_{x^d}^{d\mathrm{T}} \Delta x^d + R_{u^d}^{d\mathrm{T}} \Delta u^d + \frac{1}{2} \Delta u^{d\mathrm{T}} R_{u^d u^d}^d \Delta u^d) \Big),$$

where $R,\ G,\ R^d,\ G^d,\ L$ are the constructions of sufficient optimality conditions and $\Delta u = u - u^{\mathrm{I}},\ \Delta u^d = u^d - u^{d\mathrm{I}},\ \Delta x = x - x^{\mathrm{I}},\ \Delta x^d = x^d - x^{d\mathrm{I}}$, $m^{\mathrm{I}} = (u^{d\mathrm{I}}, x^{\mathrm{I}}, u^{d\mathrm{I}}, x^{d\mathrm{I}})$ is the given element from D.

Suppose that matrices R_{uu} and $R_{u^d u^d}^d$ are negative definite (we can always make it so by varying α [?gurman_rasina]) and find the minimum of ΔL_α with respect to $\Delta u,\ \Delta u^d,\ \Delta x,\ \Delta x_F^d,\ \Delta x^d$. We specify the functions φ, φ^d

$$\varphi = \psi^{\mathrm{T}}(k)\, x(k) + \frac{1}{2} x^{\mathrm{T}}(k)\, \sigma(k)\, x(k) + x^0,$$
$$\varphi^d = \psi^{d\mathrm{T}}(k,t)\, x^d(k,t) + \frac{1}{2} x^{d\mathrm{T}}(k,t)\, \sigma^d(k,t)\, x^d(k,t) + x^{d0},$$

where $\psi,\ \psi^d$ are the vector functions, $\sigma,\ \sigma^d$ are matrices of the corresponding dimensions.

Taking into account these requirements, we get

$$\Delta u = -R_{uu}^{-1} R_u, \qquad \Delta u^d = -R^{d\,-1}_{u^d u^d} R_{u^d}^d, \tag{1}$$

$$G_{x_F} = 0, \quad R_x = 0, \quad G_x^d = 0, \quad G_{x_F^d}^d = \mathrm{const}, \quad R_x^d = 0, \quad R_{x^d}^d = 0. \tag{2}$$

The condition (**??**) may be detailed by

$$G_{x_F} = \alpha(l + \kappa x_F) + \psi(k_F) + \sigma(k_F) x_F = 0, \tag{3}$$

$$\begin{aligned} R_x = {} & A^{\mathrm{T}} \psi(k+1) + \frac{1}{2}(Ax+B)^{\mathrm{T}} \sigma(k+1) A + \\ & + \frac{1}{2} A^{\mathrm{T}} \sigma(k+1)(Ax+B) + ax - \psi(k) - \sigma(k) x = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} G_x^d = {} & -\psi(k+1) - \frac{1}{2}\sigma(k+1)(x + \zeta(k) x_F^d) - \\ & - \frac{1}{2}(x + \zeta(k) x_F^d)^{\mathrm{T}} \sigma(k+1) + \psi(k) + \sigma(k) x - \xi^{\mathrm{T}} \psi^d - \\ & - \frac{1}{2} \xi^{\mathrm{T}} \sigma^d(t_I) \xi x - \frac{1}{2} (\xi x)^{\mathrm{T}} \sigma^d(t_I) \xi - \xi^1 x = 0, \end{aligned} \tag{5}$$

$$G^d_{x^d_F} = -\theta(k)^{\mathrm{T}}\psi(k+1) - \frac{1}{2}\zeta(k)^{\mathrm{T}}\sigma(k+1)(x+\zeta(k)x^d_F) - \\ - \frac{1}{2}(x+\zeta(k)x^d_F)^{\mathrm{T}}\sigma(k+1)\zeta(k) - \\ - \beta x^d_F + \psi^d(t_F) + \sigma^d(t_F)x^d_F = 0, \tag{6}$$

$$R^d_{x^d} = A^{d\mathrm{T}}\psi^d(k,t+1) + \frac{1}{2}(A^d x^d + \\ + B^d)^{\mathrm{T}}\sigma^d(k,t+1)A^d + \frac{1}{2}A^{d\mathrm{T}}\sigma^d(k,t+1)(A^d x + B^d) + \\ + a^d x^d - \psi^d(k,t) - \sigma^d(k,t)x^d = 0. \tag{7}$$

The equalities (**??**)–(**??**) are valid if

$$\psi(k) = A(k)^{\mathrm{T}}\psi(k+1) + \frac{1}{2}A(k)^{\mathrm{T}}\sigma(k+1)B(k) + \\ + \frac{1}{2}B(k)^{\mathrm{T}}\sigma(k+1)A(k), \quad \psi(k_F) = -\alpha l, \tag{8}$$

$$\sigma(k) = A(k)^{\mathrm{T}}\sigma(k+1)A(k) + a(k), \quad k \in \mathbf{K}\setminus\mathbf{K}', \quad \sigma(k_F) = -\alpha\kappa, \tag{9}$$

$$\psi(k) = \psi(k+1) + \frac{1}{2}\sigma(k+1)\beta(k)x^d_F + \frac{1}{2}(\beta(k)x^d_F)^{\mathrm{T}}\sigma(k+1) + \\ + \xi^{\mathrm{T}}\psi^d(k,t_I), \sigma(k) = \sigma(k+1) + \xi^{\mathrm{T}}\sigma^d(t_I)\xi + \xi^1, \quad k \in \mathbf{K}', \tag{10}$$

$$\psi^d(k,t) = A(k,t)^{d\mathrm{T}}\psi^d(k,t+1) + \\ + \frac{1}{2}A^d(k,t)^{\mathrm{T}}\sigma^d(k,t+1)B^d(k,t) + \\ + \frac{1}{2}B^d(k,t)^{\mathrm{T}}\sigma^d(k,t+1)A^d(k,t), \\ \psi^d(k,t_F) = \zeta^{\mathrm{T}}\psi(k+1) = H_{x^d F}, \tag{11}$$

$$\sigma^d(t_F) = \zeta^{\mathrm{T}}\sigma(k+1)\zeta + \beta, \quad k \in \mathbf{K}', \\ \sigma^d(k,t) = A^d(k,t)^{\mathrm{T}}\sigma^d(k,t+1)A^d(k,t) + a^d(k,t). \tag{12}$$

Note that system (**??**)-(**??**) is linear, i.e., it is certainly feasible. Denote

$$H(k,x,u,\psi(k+1)) = \\ = \psi^{\mathrm{T}}(k+1)(A(k)x(k) + B(k,u)) - b(k) - \frac{1}{2}(1-\alpha)|\Delta u|^2, \\ k \in \mathbf{K}\setminus\mathbf{K}'\setminus k_F,$$

$$H^d(k,t,x,x^d,u^d,\psi^d(k,t)) = \\ = \psi^{c\mathrm{T}}(k,t)(A^d(k,t)x^d + B^d(k,t,u^d)) - \\ - b^d(k,t,u) - \frac{1}{2}(1-\alpha)|\Delta u^d\,(k,t)\,|^2.$$

Then

$$R_u = H_u + \frac{1}{2}B_u^{\mathrm{T}}A\sigma x + \frac{1}{2}x^{\mathrm{T}}A^{\mathrm{T}}\sigma B_u + \frac{1}{2}B^{\mathrm{T}}\sigma B_u + \frac{1}{2}B_u^{\mathrm{T}}\sigma B + b_u,$$

$$R_{uu} = H_{uu} + \frac{1}{2}B_{uu}^{\mathrm{T}}A\sigma x + \frac{1}{2}x^{\mathrm{T}}A^{\mathrm{T}}\sigma B_{uu} + B_u^{\mathrm{T}}\sigma B_u + B^{\mathrm{T}}\sigma B_{uu} + b_{uu},$$

$$\begin{aligned} R^d_{u^d} = H^d_{u^d} + \frac{1}{2}B^{d\mathrm{T}}_{u^d}A^d\sigma^d x^d + \frac{1}{2}x^{d\mathrm{T}}A^{d\mathrm{T}}\sigma^d B^d_{u^d} + \\ + \frac{1}{2}B^{d\mathrm{T}}\sigma^d B^d_{u^d} + \frac{1}{2}B^{d\mathrm{T}}_{u^d}\sigma^d B^d + b^d_{u^d}, \end{aligned}$$

$$\begin{aligned} R^d_{u^d u^d} = H^d_{u^d u^d} + \frac{1}{2}B^{d\mathrm{T}}_{u^d u^d}A^d\sigma^d x^d + \frac{1}{2}x^{d\mathrm{T}}A^{d\mathrm{T}}\sigma^d B^d_{u^d u^d} + \\ + B^{d\mathrm{T}}_{u^d}\sigma^d B^d_{u^d} + B^{d\mathrm{T}}\sigma^d B^d_{u^d u^d} + b^d_{u^d u^d}. \end{aligned}$$

Note that from the equality $R^d_x = 0$ we can't obtain equations. Obviously, these formulas for first and second derivatives of R, R^d with respect to control variables linearly depend on state variables of the upper and lower levels, respectively. Therefore, the obtained solution represents approximate optimal control synthesis.

4. Iterative procedure

As a whole, we get the following iterative procedure on a step s.

1. «Left to right» we compute NDS for $u = u_s(k)$, $u^d = u^d_s(k,t)$ with given initial conditions, getting the corresponding trajectories $x_s(k)$, $x^d_s(k,t)$.
2. «Right to left» we resolve NDS with respect to $\psi(k)$, $\psi^d(k,t)$, $\sigma(k)$, $\sigma^d(k,t)$ according to (**??**)–(**??**).
3. We find Δu, Δu^d and new controls $u_{s+1}(k) = u_s(k) + \Delta u$, $u^d_{s+1}(k,t) = u^c_s(k,t) + \Delta u^d$ according to (**??**).
4. «Left to right» we calculate the initial NDS with the controls found and given initial conditions.

This iterative process is over when $|I_{s+1} - I_s| \approx 0$ with a given precision.

Theorem 3. *Suppose that the formulated iterative procedure is constructed for a given NDS, and functional I is bounded from below. Then it generates the improving sequence $\{m_s\} \in \mathbf{D}$ that converges with respect to the functional, i.e., there is a number I^* such that $I^* \le I(m_s)$, $I(m_s) \to I^*$.*

Proof. The proof immediately follows from the monotone property of the improvement operator under consideration. Thus, we get a monotone sequence of numbers

$$\{I_s\} = \{I(m_s)\}, \quad I_{s+1} \le I_s,$$

bounded from below, that converges to a certain limit $I_s \to I_*$. $\square$

5. Example

Consider the following NDS

$$\begin{aligned}
x^{d0}(t+1) &= x^{d0}(t) + \frac{1}{2}(x^d(t))^2 + \frac{1}{3}(u_1^d)^3,\\
x^d(t+1) &= -2x^d(t) + (u_1^d - 1)^2, \quad x^d(0) = 1, \qquad t = 0,1,2,3,\\
x^{d0}(t+1) &= x^{d0}(t) + \frac{1}{2}(x^d)^2 + u_2^d,\\
x^d(t+1) &= (t - u_2^d)^2, \qquad t = 4,5,6,\\
I &= x^d(7) \to \min.
\end{aligned}$$

It is easy to see that $K = 0,1,2$. Since x^d is a linking variable in the two periods under consideration, we can write a process of the upper level in terms of this variable $x(0) = x^d(0,0)$, $x(1) = x^d(0,4)$, $x(2) = x^d(1,7)$, $x(1) = x^d(0,4)$, and $x^d(1,4) = x(1)$. Then $I = x(2)$. From this it follows $a^d(0,t) = 1$, $b^d(0,t) = \frac{1}{3}(u_1^d)^3$, $A^d(0,t) = 1$, $B^d(0,t) = (u_1^d - 1)^2$, $a^d(1,t) = 1$, $A^d(1,t) = 0$, $B^d(1,t) = (t - u_2^d)^2$, $\zeta = 1$, $\xi = 1$, $\xi^1 = 0$, $\beta = 0$.

The equations of the method have the form

$$\begin{aligned}
&\psi^d(0,t) = -2\psi^d(0,t+1) - 2\sigma^d(0,t+1)(u_1^d - 1)^2,\\
&\qquad\qquad\qquad\qquad \sigma^d(0,t) = 4\sigma^d(0,t+1) + 1,\\
&\psi^d(1,t) = 0, \qquad\qquad \sigma^d(1,t) = 1,\\
&\psi^d(1,t_F) = \psi(k_F) = -\alpha, \qquad \sigma^d(1,t_F) = 0,\\
&\psi(2) = -\alpha, \qquad\qquad \sigma(2) = 0,\\
&\psi(1) = \psi(2) + \frac{1}{2}\sigma(2), \qquad \sigma(1) = \sigma(2) + \sigma^d(1,t_I),\\
&\psi^d(0,t_F) = \psi(1), \qquad\qquad \sigma^d(0,t_F) = \sigma(1),
\end{aligned}$$

$$\begin{aligned}
R^d(0,t)_{u_1^d} &= 2\psi^d(0,t+1)(u_1^d - 1) +\\
&+ 2\sigma^d(0,t+1)((u_1^d - 1)^2 - 2x^d)(u_1^d - 1) + (u_1^d)^2 - (1-\alpha)\Delta u_1^d,\\
R^d(0,t)_{u_1^d u_1^d} &= 2\psi^d(0,t+1) + 6\sigma^d(0,t+1)(u_1^d - 1)^2 -\\
&- 4\sigma^d(0,t+1x^d + 2u_1^d - (1-\alpha),\\
R^d(1,t)_{u_2^d} &= -2\psi^d(1,t+1)(t - u_2^d) - 2\sigma^d(1,t+1)(t - u_2^d)^3 +\\
&+ 1 - (1-\alpha)\Delta u_2^d,
\end{aligned}$$

$$R^d(1,t)_{u_2^d u_2^d} = 2\psi^d(1,t+1) + 6\sigma^d(1,t+1)(t-u_2^d)^2 - (1-\alpha).$$

The initial approximation for the improvement procedure is $u^d(0) = u^d(1) = 1$, $u^d(2) = u^d(3) = 2$, $u^d(4) = -1$, $u^d(5) = -1$, $u^d(6) = -1$. The results are given by Fig. **??** and Table **??**.

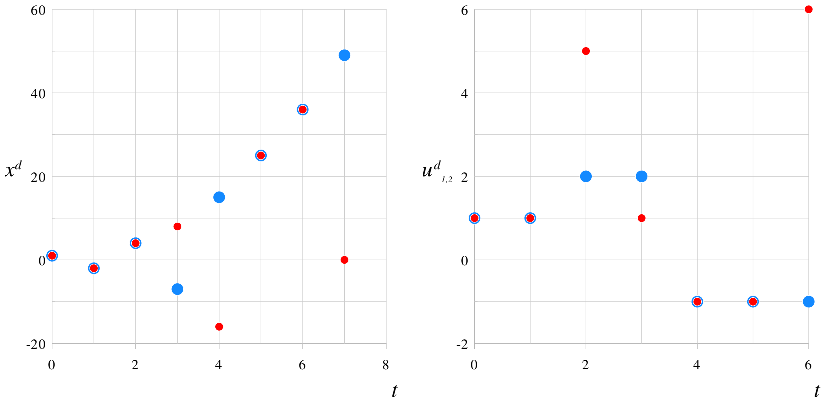

FIGURE 1. The initial (blue) and final (red) iterations of the method

TABLE 1. Changes of the functional on iterations

Iteration	Functional
0	49
1	16
2	4
3	0

Conclusions

In this paper, we propose an approximate optimal control synthesis method for nonuniform discrete systems linear quadratic with respect to state. It is derived from the well-known scheme when specifying Krotov function. We develop an algorithm that implements this method and demonstrates its quality through a model example.

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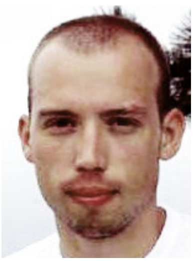

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