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Methods for anchoring boundary nodes when smoothing a triangular surface mesh

Abstract. In numerical modeling tasks that use surface meshes, remeshing is often required. However, while remeshing, distortion can occur. The accumulation of distortions can lead to the collapse of the solution. Smoothing algorithms are used to maintain the quality of the mesh during the calculation. When performing smoothing using methods that shift the mesh nodes, the border nodes are usually fixed to avoid distortion. However, simply fixing the nodes can lead to more severe distortion. This paper presents methods for working with boundary nodes to control such nodes during the smoothing process. Algorithms for working with pseudo-3D surface meshes, which are of particular interest, are also considered.

Key words and phrases: computational geometry, smoothing, triangular mesh, numerical modeling.

2020 *Mathematics Subject Classification*: 65N50; 68N19,74A50

Introduction

This article discusses computational geometry problems concerning surface unstructured triangular meshes. Such meshes are often used in issues of numerical modeling and computational geometry.

In the process of developing programs for numerical modeling, a problem arises: mesh smoothing techniques (Laplace, Taubin smoothing [1], methods that preserve mesh features [2–4]), turn out to be unsuitable for working with meshes that are not closed surfaces. Boundary nodes move towards the body, causing uncontrolled compression of the surface. This feature is unacceptable since the task of preserving the boundaries of the mesh and its geometric features is critical.

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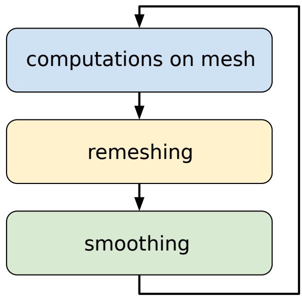


FIGURE 1. Iterative remeshing

1. Smoothing methods

Smoothing surface meshes are used to improve the quality of the mesh. A high-quality mesh should describe the surface of the body well and at the same time consist of high-quality faces.

In some numerical modeling tasks, such as calculating body icing [5] or calculating thermal expansion [6], an iterative rebuilding of the surface occurs (Figure 1).

While remeshing, as the nodes move, their clusters arise. In some places, the mesh becomes sparse, and in others, it becomes denser. This grid configuration negatively affects the subsequent iterations of the calculation. Various smoothing methods are used to maintain the mesh quality throughout the computation.

There are many techniques for smoothing surface meshes. This article discusses triangular meshes and methods that use node shift. In numerical modeling using the finite element method, it is necessary to preserve the mesh topology.

The task of moving a node can be formulated as follows:

$$p_i \leftarrow p_i + \lambda p'_i, \tag{1}$$

where p_i, $p'_i \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$ — displacement factor.

Classic Laplace smoothing is

$$p'_i = \frac{1}{|N_i|} \sum_{p_j \in N_i} (p_j - p_i),$$

where $N_i = \{p_j : (p_i, p_j) \in E\}$ and E — set of mesh edges.

1.1. Null-space smoothing

Null Space Smoothing (NSS) [7] is used to smooth surface meshes in case volume is preserved. The null space is a tangent plane for smooth mesh regions, a tangent line for edges, and an empty set for corners. These spaces are formed proceeding from the analysis of the eigenvectors obtained by the spectral decomposition of the matrix formed for each grid node:

$$A = N^T W N.$$

Here N is a matrix of size $m \times 3$ containing normals of faces adjacent to the node, m is a number of faces adjecent to the node, W is a diagonal matrix of weights of size $m \times m$, where W_{ii} is a weight, corresponding to the i-th face.

Since A is a symmetric positive definite matrix, there is a spectral decomposition

$$A = V \Lambda V^T,$$

where Λ is a diagonal matrix, containing eigenvalues of A. The eigenvectors e_i correspond to $\lambda_i = \Lambda_{ii}$ form the columns of matrix V.

As a result, for each node we have three eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Take some threshold λ' (it becomes user-defined parameter), we compare with it eigenvalues we obtained and choose those that exceed it. Having chosen some threshold value λ' (it becomes a tunable parameter), we compare the existing eigenvalues with it and select those that exceed it.

The primary space

$$V_{prim} = \{e_i \mid \lambda_i > \lambda'\}$$

is formed from eigenvectors that correspond to the following eigenvalues.

The orthogonal complement to this space is null space.

Depending on the dimension k of the primary space, the nodes belong to one of three classes: a node either lies in a plane, lies on edge, or a corner node. Spectral decomposition allows us to determine the spatial configuration of the neighborhood of the node.

Null-space is formed the next way

$$V_{null} = \begin{cases} \{e_2, e_3\}, & \text{if } k = 1 \\ \{e_3\}, & \text{if } k = 2 \\ \varnothing, & \text{if } k = 3, \end{cases}$$

where V_{null} is a null-space and k is the dimensionality of the primary space.

Shift vector of the node

$$dv = \frac{\sum_{j=1}^{m} w_j c_j}{\sum_{j=1}^{m} w_j}. \tag{2}$$

is calculated over a number m of adjacent faces, a non-negative weight $w_j \in \mathbb{R}$ of j-th face, and a centroid c_j of j-th face.

Project shift vector dv on the null-space:

$$p' = \begin{cases} s_t \sum_{i=k+1}^{3} (dv \cdot e_i) e_i, & \text{if } k = 1 \text{ or } k = 2 \\ 0, & \text{if } k = 3, \end{cases}$$

here $s_t > 0 \in \mathbb{R}$ is a regulation of magnitude of shift and e_i is an eigenvector.

1.2. Fuzzy Vector Median-Based Surface Smoothing

Laplace smoothing, Taubin smoothing, and similar methods do not preserve the geometric features of mesh [8]. In a volume conservation task, it is necessary to maintain all possible geometric features, such as corners and edges, while allowing the nodes to move only along the surface.

The Fuzzy Vector Median (FVM) [9] smoothing method consists of filtering face normals and then moving the mesh nodes according to the updated normals. Filtering is done using median filter.

A median normal is a normal from a set of face's normals that deviates at the smallest angle from all other normals;

$$x_{VM} = \arg\min_{x \in \Omega} \sum_{i=1}^{N} \|x - x_i\|,$$

where Ω is a set of faces, $N = |\Omega|$, x_i is a normal of the i-th face. This movement leads to severe distortion at the edges of the mesh, which can destroy the solution.

The smoothed face's normal f is computed using the weighted average of all median normals adjacent to f.

$$x_{FVM} = \frac{\sum_{i=1}^{N} x_i \ \mu_G (x_i, x_{VM})}{\sum_{i=1}^{N} \mu_G (x_i, x_{VM})},$$

Here $\mu_G(u, v)$ is a Gaussian membership function

$$\mu_G(u, v) = e^{-\|u-v\|/2\sigma^2},$$

where $\sigma > 0 \in \mathbb{R}$ — spread parameter.

The nodes are shifted to minimize the difference between the current face normal and the resulting normal

$$p_i \leftarrow p_i + \lambda \sum_{j \in i^*} \sum_{f \in F_{ij}} n_f n_f^\top (x_j - x_i), \quad i = 1, 2, \cdots, V,$$

using the set $i^* = \{j : \{i, j\} \in E\}$, faces F_{ij} adjoin with the edge (i, j), normal n_f to the face f, and i-th node x_i of the grid.

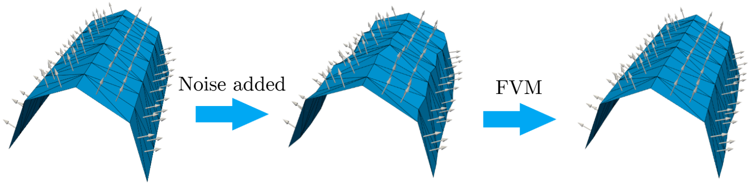


FIGURE 2. Application of fuzzy vector median-base surface smoothing

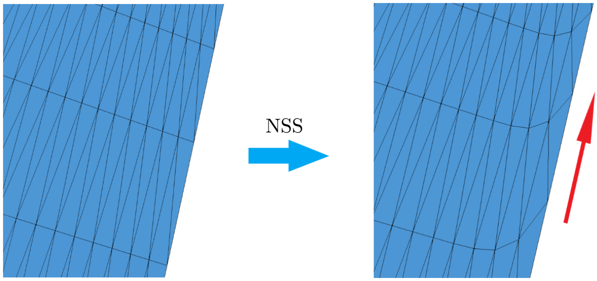


FIGURE 3. Mesh structure before and after smoothing with volume preserving

Figure 2 shows an example of applying the fuzzy vector median-based method to a surface mesh. Arrows indicate face normals. Random noise was introduced into the nodes of the original mesh. The method restores the original surface in 20 iterations.

2. Formulation of the problem

Figure 3 shows the original mesh structure near the border before and after smoothing using the NSS algorithm. This uniform mesh structure often occurs when triangulating a pseudo-3D surface, such as a wing profile of particular interest.

Since a node must move in the direction of the mean of centroids of adjacent faces, even if all other nodes in the mesh stop moving during the smoothing process, the border nodes still move along the edge. The nodes have more adjacent faces in one direction, and those faces are pulling them off. This movement leads to severe distortion at the edges of the mesh, which can destroy the solution.

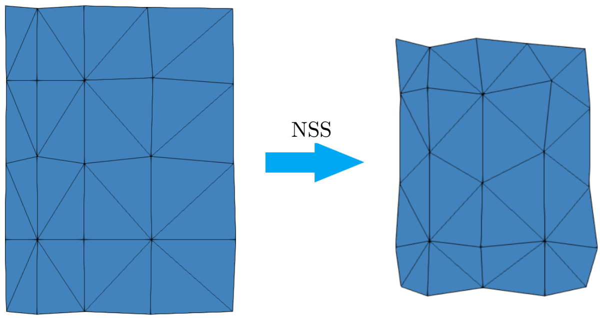


FIGURE 4. An example of mesh distortion after 10 iterations of null-space smoothing

3. Triangular mesh quality criterion

A common criterion for the quality of triangular meshes is the coefficient $\alpha \in [0, 1]$ [10]:

$$\alpha_i = \frac{4\sqrt{3}A_i}{||P_1P_2||^2 + ||P_2P_3||^2 + ||P_3P_1||^2},$$

depending on a square A_i of i-th face and coordinates P_1, P_2, P_3 of face's nodes.

For an equilateral triangle $\alpha = 1$. The geometric mean α_{mean} and the minimal value α_{min} are used to estimate the entire surface.

4. Methods of controlling boundary nodes

In smoothing the surface of a mesh, it is crucial to preserve its area and geometric shape. For the nodes to move only along the body's surface, the previously described smoothing methods preserving the volume are applied. These methods are effectively applied to grids with a closed volume (watertight). However, if the mesh is not closed and there are border nodes in it, then any of the methods using the shift of nodes will change the border of the mesh (Figure 4). Therefore, boundary nodes are often fixed. Fixing the boundary leads to distortion if, for example, the nodes in the middle of the mesh are actively shifted. The challenge is to keep the boundary nodes moving without changing the mesh boundaries or making user-controlled changes. It is also necessary to maintain acceptable mesh quality.

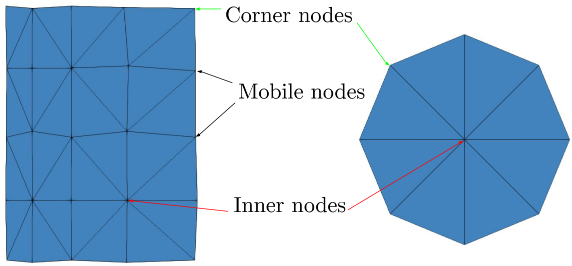


FIGURE 5. Defined node types

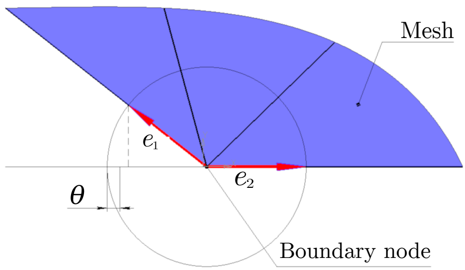


FIGURE 6. User defined θ coefficient

Let's introduce the notation for the nodes (Figure 5).

Inner — node that doesn't have boundary edges.

Boundary — node that is adjacent to boundary edge.

Corner — boundary node, that is fixed on the current iteration of smoothing.

Mobile — boundary, but not corner.

In order to control the boundaries of the mesh in the process of smoothing by methods with the preservation of volume, the following technique is proposed.

4.1. Boundary nodes move along edges only

Let's fix the movement of a mobile node only along an adjacent boundary edge. We represent two of its adjacent boundary edges in the form of unit vectors e_1 and e_2 (Figure 6),

$$0 \leq \theta \leq 2, \;\; \theta \in \mathbb{R}.$$

Having received the displacement vector of the node, we calculate its projection onto e_1 or e_2 and choose a larger value to allow the node not to

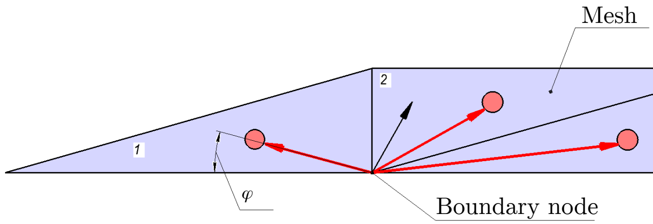


FIGURE 7. Example of unsuccessful arrangement of adjacent faces. The centroids of adjacent faces are marked in red. Black - the vector of displacement of the boundary node

move towards the body and keep its boundaries:

$$p'_i \leftarrow \begin{cases} e_1(p'_i \cdot e_1), & \text{if } p'_i \cdot e_1 \geq p'_i \cdot e_2 \\ e_2(p'_i \cdot e_2), & \text{in opposite case.} \end{cases}$$

4.2. Fixing nodes

Corner nodes, even when moving along the edges, significantly change the surface area of the mesh. Therefore, we introduce the coefficient $\theta \in \mathbb{R}$, which determines the rigidity of fixation of such nodes.

The user defines a relation that controls the permissible projection length of e_1 on e_2. The coefficient θ determines whether the algorithm will fix the node p_i to conserve area.

The node p_i stays fixed:

$$p'_i = 0 \Leftrightarrow e_1 \cdot e_2 \geq \theta - 1.$$

The larger the θ parameter, the more the algorithm tends to define the node as a corner and fixes its position. Figure 5 shows two grids. The mesh on the left has four corner nodes; all the rest can be moved along the boundary edges for smoothing. The faces on the right are more extensive, and the smoothing algorithm will tend to move the nodes to the right to align the sizes of the triangles. All boundary nodes are corner nodes on the right grid and are therefore fixed during the smoothing process.

4.3. Weight function for centroids

Consider the case when adjacent faces for the boundary node are located as shown in Figure 7.

If a similar configuration is maintained throughout the entire boundary of the body (Figure 3), then the nodes will move at each iteration, following the direction of displacement, which will lead to distortions.

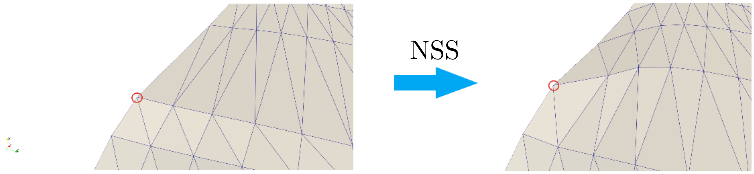


FIGURE 8. Mesh distorting in movable nodes

To exclude such a movement, the weights are used when calculating node offset to neutralize the effect of adjacent faces whose centroids located close to the border of the body.

Consider the boundary node p_i. There are various methods for setting the w_j weight for the j-th adjacent face in the formula (2). This work use the following setting of weights

$$w_j = \begin{cases} 1 - |\cos(\varphi_j)|, & \text{if } p_i \text{ — mobile node} \\ 1, & \text{otherwise,} \end{cases}$$

where φ_j — angle in radians between the vector from p_i to the centroid of face j and an boundary edge, adjacent with this node, as shown in Figure 7.

The closer the angle φ to $\pi/2$, the larger weigths the face j gets. Thus, the node moves along the boundary with a smaller step.

4.4. Limiting the magnitude

For user control, it makes sense to artificially underestimate the magnitude of movement of the movable node. Let's define coefficient β in the formula (1) so that $\beta \ll \lambda$:

$$p_i \leftarrow \begin{cases} p_i + \lambda p_i', & \text{if } i \text{ — inner node} \\ p_i + \beta p_i', & \text{otherwise.} \end{cases}$$

4.5. Application of FVM when smoothing a pseudo-3D profile

Applying NSS to pseudo-3D profiles produces distortion, especially at the edges. These distortions arise due to many factors including the features of NSS algorithm, as well as the methods presented. Figure 8 shows an example of such a distortion.

The resulting distortion of the grid can negatively affect the calculation process in the future. During the next iteration of smoothing, the defect can only increase. Over time, such problems will accumulate, and the calculation process may end up with an error.

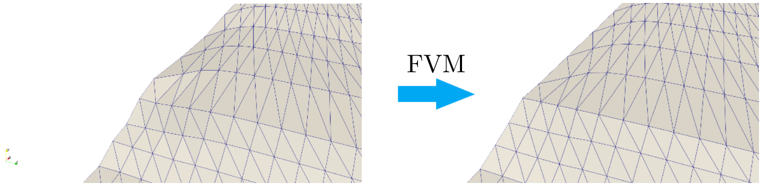


FIGURE 9. Example of FVM smoothing

To solve such problems, FVM was applied. Its expected effect on the pseudo-3D profile is to smooth out the arising irregularities and return the mesh to the pseudo-3D state. To save resources, it may seem attractive to perform smoothing only on moving nodes - where distortions occur. However, experiments show that there are so many special cases in the geometry of grids that the positive effect of the saved computational resources is lost. Distortion can occur at various locations in the mesh, and therefore the optimal strategy is to **smooth the entire mesh**.

Fuzzy vector medians and normals obtained on their basis contain information about the shape of the surface. Thus, the nodes move to the plane of adjacent faces (Figure 9).

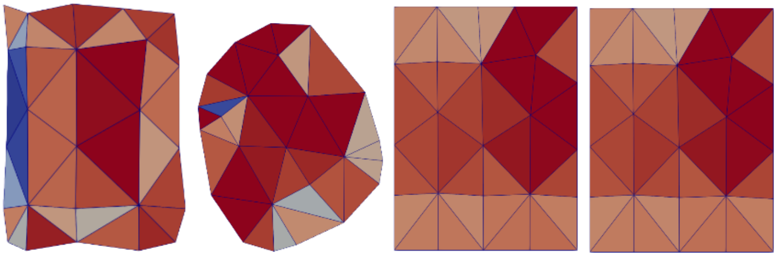
Despite the fact that FVM smoothing affects the volume, the algorithm parameters allow you consider the shape of the mesh. For example, when working with pseudo-3D profiles, it makes sense to use the set $\{f_j : f_j \cap f_i \in V\}$ for the face of interest : f_j, since usually more than half of neighboring faces lie in the same plane with the face in question. This makes its vector median equal to the prevailing normal among neighboring faces, and therefore the algorithm aligns the node's position in a single plane.

5. Experimental results

Consider the results of applying the methods presented in 4.1 - 4.4 to the mesh from Figure 4. The smoothing algorithm is NSS.

As you can see from the Figure 10, the presented technique allows obtaining high-quality mesh smoothing.

The color shows the value of the quality metric α. The combination of the presented methods allows you to preserve the grid's boundaries while aligning the size of the faces. Weight function for the centroids and fixing of the nodes does not increase the quality of this mesh.



(*a*) Laplace smoothing only, $\alpha_{mean} = 0.672$, $\alpha_{min} = 0.366$

(*b*) nodes move only along edges, $\alpha_{mean} = 0.770$, $\alpha_{min} = 0.014$

(*c*) node fixation, $\theta = 0.2$, $\alpha_{mean} = 0.877$, $\alpha_{min} = 0.79$

(*d*) weight function for the centroids and fixing of the nodes

FIGURE 10. Consistent application of the proposed methods

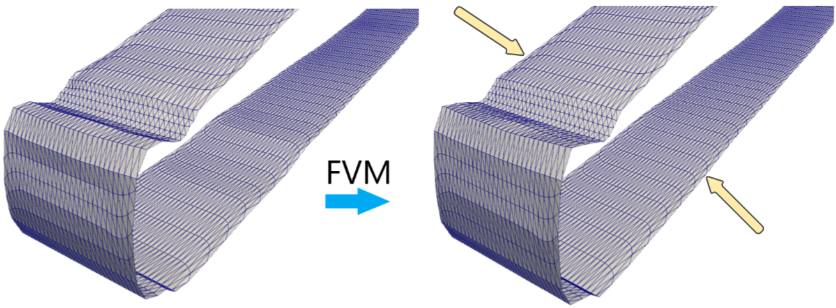


FIGURE 11. Example of FVM smoothing. 50 iterations, $\sigma = 0.1, \lambda = 0.05$. Arrows show the places where nodes align

Figure 11 shows an example of applying the presented methods and smoothing based on fuzzy medians to a large mesh. FVM removes post-NSS distortion. This allows the mesh to return to a pseudo-3D state.

Conclusion

The article discusses working with the boundary nodes of an unstructured triangular mesh when smoothing the mesh. The proposed methods are based on the analysis of the mutual arrangement of the surface mesh elements and allow the user to control the behavior of the boundary nodes.

The smoothing method based on fuzzy vector medians allows you to eliminate distortions when working with a pseudo-3D mesh. This type of meshes has its own characteristics and is often found in problems of numerical modeling.

The results obtained allow us to conclude that the using proposed methods will allow avoiding the loss of geometric features of the mesh during the smoothing process. At the same time, boundary nodes can move and participate in the smoothing process, in contrast to the traditional approach, which consists of simply fixing them.

The proposed methods were applied in the problem of modeling the process of ice accretion n the surface of an aircraft wing. The mesh requires iterative rebuilding to reflect the shape of the accumulated ice. The proposed methods make it possible to control the boundary nodes and obtain a mesh of acceptable quality, which requires manual adjustment and control of the proposed parameters.

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